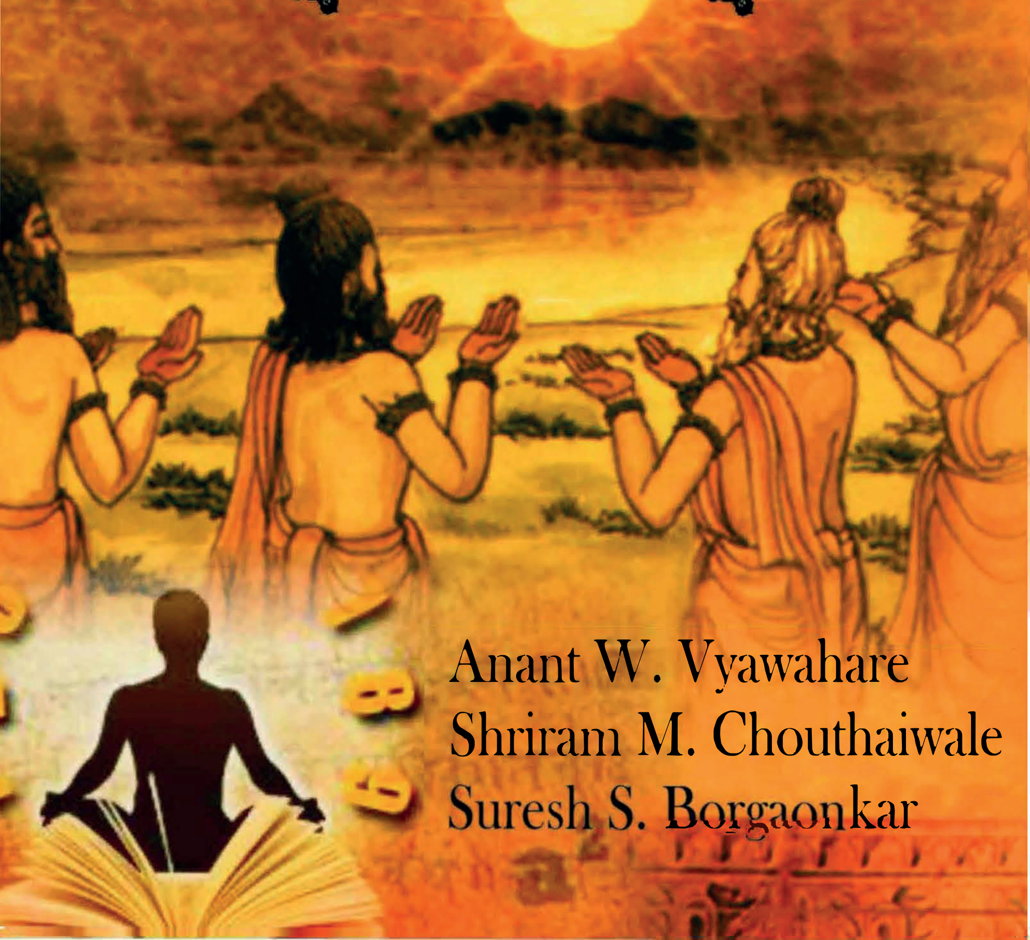
नचिकेत ई-बुक्स

Introduction to Vedic Mathematics



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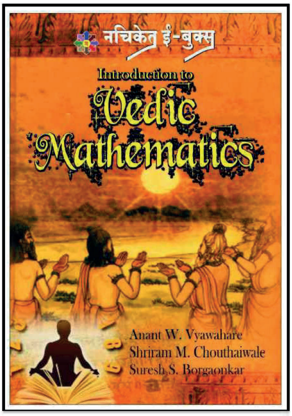
VEDIC MATHEMATIC



☯ लेखक ☯

◆अनंत डब्ल्यू. व्यवहारे ◆श्रीराम एम. चौथाईवाले

◆सुरेश एस. बोरगावकर

नचिकेत प्रकाशन

नागपूर - ४४० ०१५

Vedic Mathematic

- नचिकेत ई बुक्स क्र. ३४
- **प्रथम आवृत्ती :** ११ जुलै २०१४, आषाढ शु.१४, युगाब्द ५११६
- **कॉम्प्युटर आणि मोबाईलवर वापरण्यासाठी ई-आवृत्ती उपलब्ध.**

- **लेखक**
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Contents

LIST OF SUTRAS (FORMULAE)

(1) एकाधिकेन पूर्वेण - Ekadhiken purvena
(By one more than the previous one)

(2) निखिलम् नवतश्चरमंदशत: - Nikhilam Navatasccharamam Dashatah
(All from nine, and last from ten)

(3) ऊर्ध्वतिर्यग्भ्याम् - (Urdhva tiryagbhyam)
(Vertically and crosswise)

(4) परावर्त्य योजयेत् - Paravartya Yojayet
(Transpose and apply)

(5) शून्यं साम्यसमुच्चये - Sunyam Samyasamuchchye
(The summation is equal to zero)

(6) आनुरूप्ये शून्यमन्यत् - Anurupye Sunyamanyat
(If one is in ratio, the other is zero)

(7) संकलन व्यवकलनाभ्याम् Sankalana Vyavakalanabhyam
(By addition and subtraction)

(8) पुरणापूरणाभ्याम् - Puranapuranabhyam
(By the completion and noncompletion)

(9) चलन कलनाभ्याम् - Chalana kalanabhyam
(Sequential motion)

(10) यावदूनम् Yavadunam
(The deficiency)

(11) व्यष्टिसमष्टि: - Vyastisamastih
(Whole as one and one as whole)

(12) शेषाङ्केन चरमेण Sesanyankena
(Remainder by the last digit)

(13) सोपान्त्यद्वयमन्त्यम् Sopantyadvayamantyam
(Ultimate and twice penultimate)

(14) एकन्यूनेन पूर्वेण Ekanyunena Purvena
(By one less than the previous)

(15) गुणितसमुच्चय: Gunitasamuccayah
(The whole product is the same)

(16) गुणकसमुच्चय: Gunakasmuccayah
(Collectivity of Multipliers)

UPA - SUTRAS (Sub-Formulae)

(1) आनुरूप्येण - Anurupyena
(propotionately)

(2) शिष्यते शेषसंज्ञः – Sisyate Sesasamijnah
(The remainder is the constant)

(3) आद्यमाद्येनामन्त्यमन्त्येन – Adyamadyenantyamantyena
(First by the first and last by the last)

(4) केवलैः सप्तकंगुण्यात् - Kevalaih Saptakam Gunyat
(1/7 By the product)

(5) वेष्टनम् Vestannam
(Circumscribed)

(6) यावदूनम् तावदूनम् - Yavadunam Tavadunam
(What ever deficiency further lessen that much)

(7) यावदूनम् तावदूनीकृत्यं वर्गं च योजयेत्
Yavadunam Tavadunikrty vargam cha yogayet
(Lesser by the deficiency and use its square)

(8) अन्त्ययोर्दशकेऽपि Antyayordashakepi
(Sum of last digits is ten)

(9) अन्त्ययोरेव – Antyayoreve
(Only by the last)

(10) समुच्चयगुणितः - Samuccayagunitah
(Product of Whole)

(11) लोपनस्थापनाभ्याम् – Lopanasthapanavhyam
(By Alternate Elimination and Retention)

(12) विलोकनम् – Vilokanam
(By observation)

(13) गुणितसमुच्चयः समुच्चयगुणितः Gunitsamuccayah Samuccayagunitah
(Product of the whole is equal to whole of the product)

(14) द्वन्द्वयोग - Dwandwayoga
(Duplex)

(15) शुद्धः –Shudha
(Purify)

(16) ध्वजांकः Dhwajanka
(Flag digit)

PREFACE

This text deals with Vedic Mathematics (VM) based on the basic volume on the subject written by His Holiness Jagadguru Shankaracharya, Shri Bharati Krishna Teertha of Gowardhan Peeth, Jagannathpuri, Orissa, India.

It was his sheer intelligence & genius that he constructed the Sanskrit formulae from the Atharvaveda. Hence this mathematics is Vedic Mathematics.

Now VM is used, not only in Science, but also in the field of Technology. It has already crossed the Science boundaries of India & has became a leading subject of research abroad. VM is not only a mathematical wonder, it is logical also. It is true that usual process of induction and deduction are hidden in VM. Any result looks like a magic but proving the result is logic, VM is both ! Hence VM has a degree of eminence which cannot be disproved.

There are many books on VM. But its branches are discussed in different books. This book intends to cover all of them. This book is useful to students up to graduate level.

This motivated us to write this book and we tried to assemble the subject in an easy possible manner.

We do not claim for the originality of this text. The text of VM by Swamiji (Pub. Motilal Banarasidas, New Delhi, India) is a guidelamp & reference book. We are thankful to the publishers of this book.

We thank Mr. Wijay Chitaley, Managing Director, All India Reporter Pvt. Ltd., Congress Nagar, Nagpur for his generous donation of paper required for this text.

We are thankful to Mr. R. M. Pujari, President G.G. Joshi Shilp Sanshodhan Pratisthan, Nagpur, for going through the manuscript and making valuable suggestions. & for his article on 'Ancient Hindu Mathematics'.

We request all students and admirers of Mathematics & Applied Sciences to go through this text. Answers to problems in exercises are given only when necessary. We hope this text will inspire one & all to practice VM.

Suggestions, in any form, are welcome.

We express our gratitude to Mr. Wijay Chitaley, Managing Director, All India Reporter Pvt. Ltd., Congress Nagar, Nagpur for neat printing & Swadhyaya Mandal, Killa Pardi, Gujarat for publishing this text.

Thanks.

AWV

June 2003 SMC

Nagpur, India. SSB

(Authors)

Jagadguru Swami Shri Bharati Krishna Tirthji

A Pioneer of Vedic Mathematics.

His Holiness Shri Swami Bharati Krishna Tirth, better known by his disciples by name Gurudev or Swamiji, was born in March 1884 at Tennivelly (Tamilnadu, India). His father Shri P. Narsimha Shastri was Deputy Collector.

Swamji, named as Venkataraman in his early days, was very brilliant and genius. He passed matriculation in 1899 from Madras University with the topmost position.

He was proficient in Sanskrit Oratory. Hence he was awarded the honour "Saraswati" by Madras Sanskrit Association in 1899 at the age of 15. After winning the highest place in B.A. examination, Swamiji, then Venkataraman Saraswati, appeared at M.A. examination of the American College of Science, Rochester, New-York from Bombay centre in 1903. In 1904 at the age of 20, he passed M.A. Exams in Sanskrit, Philosophy, English, Mathematics, History & Science & secured the first position at the same time in all subjects. What a record of academic brilliance ! Later on, he became the principal of National College, Rajmahendri. But in 1911, he could not resist his burning desire of spiritual knowledge & practice. He resigned from the post and joined Sringeri Matha. There he studied Vedanta philosophy & practised Brahma Sadhana.

After the marriage of his daughter and demise of his wife, he was initiated into the holy order of Sanyas at Varanasi by His Holiness Jagadguru Shankaracharya Shri Trivikram Tirth of Sharda peeth of Shrigeri on 4th July 1919. On this occasion he was given the new name, Swami Bharati Krishna Tirth. In 1921, he was installed at the throne of Jagadhuru

Shankaracharya of Sharda Peeth, Shringeri. In 1925, he became the Shankaracharya of Govardhan Peeth of Jagannath Puri, Orissa, India.

In Shringeri he studied all the Vedas. After research in Atharvaveda, he constructed sixteen mathematical formulae in Sanskrit. Obviously these formulae are not to be found in the present Text of Atharvaveda. They were actually constructed on the basis of intuitive revelation from the materials scattered in Atharvaveda.

Later on, he constructed fourteen more formulae. These 30 formulae, in all, form the basis of Vedic Mathematics (VM). He afterwards wrote sixteen volumes on VM. Unfortunately these volumes were lost in 1956. Even then Swamiji was not perturbed as all these volumes were stored in his memory.

He delivered lectures on VM at Calcutta in presence of selected audience including well known mathematicians. At this instance, he was invited to lecture on VM in USA. In this tour he rewrote, from memory, a volume on VM which is presently available.

In 1953, he founded an institution named 'Shri Vishwa Punarnirman Sangha' at Nagpur, (India) to promote VM, spiritual ideas and humanitarian services. After his return in 1958 from USA and U.K. he delivered a series of lectures on VM in the premises of Nagpur University, which was largely attended by elite of society including mathematicians. From this instant, VM became known to the academic world. He visited many universities in India and propogated VM.

Swamiji had been undergoing a terrific strain for more than five decades in devoting his body, mind, heart to the cause of service and humanity and revival of Vedantic ideas. His health

deteriorated and as a result he fell seriously ill in 1959. Finally he breathed last and took Mahasamadhi at Mumbai on 2nd Feb. 1960.

Many great personalities were among his disciples. These include, Dr. Rajendraprasad, the first President of India, Ex. Chief Justice B.P. Sinha, then Finance minister & well known scholar in Sanskrit Dr. C.D. Deshmukh, Dr. V.S. Agrawal & Dr. Premlata Sharma, all professors of Banaras Hindu University, India.

His winning personality, charming innocence, eager thirst of knowledge, religious zeal, belief in Vedas & Shastras, universal kindness, photographic memory were the virtues which worked as a magnet.

He was a poet of uncommon merit. Praising the Guru & The Almighty, his book "Sanatana Dharma", published by Bharatiya Vidya Bhavan, Mumbai, is well celebrated.

Swamiji was a great Yogi, a high ranking personality and an ace mathematician. He was an admirer of a great poet Bhartrihari. He was a gifted scholar and a true Sanyasi.

He was a prolific writer and eloquent speaker in English & Sanskrit also. Swamiji used to call VM as "mental" mathematics.

The VM stated by Swamiji is not only magical, it is logical too. Full of short tricks & methods, VM deals will all basic branches of mathematics.

May all the world benefit by of his life, so nobly, so spiritually & lovingly. May Swamiji's holy spirit shower on the world.

Ancient Hindu Mathematics

India is the home of ancient-most civilisations and the Vedas are the storehouse of knowledge. These statements carry with them one message that many scientific ideas must have attracted the thinkers of another closed-by civilisations. Thus the history of science and technology gets drafted. The history with which we are familiar at present is the contribution of Romans, Egyptians, Babylonians, Greeks and Greco-Romans. The contributions of Chinese sciences are now coming forth. Unless systematic studies of Indian contribution to the history of science and technology is taken up in the ramification and dynamics of scientific ideas, the present current history cannot be but an incomplete picture.

Historically the host of Indian scientific ideas have influenced the rest of the world. The knowhow of mathematics in India was of a very high order. "Ganeeta" occupied a subject of prime importance amongst Indian intellectuals. Numbers had a special appeal to the Indians of Vedic and post-Vedic periods. Yajurved Samhita and Panchvinsha Brahman contains the terminology of numerals in ascending decimal scale l. e.

एक, दश	(10^1)				
शत	(10^2)	सहस्त्र	(10^3)	मयुत	(10^4)
नियुत	(10^5)	अयुत	(10^6)	अर्बुद	(10^7)
न्यर्बुद	(10^8)	समुद्र	(10^9)	मध्य	(10^{10})
जत	(10^{11})	परार्ध	(10^{12})		

It is interesting to note that the Greeks knew the highest term in fourth century B. C. as 10^4 (Myriad) and contemporarily Romans knew 10^3 as Mille. The Indians on the other hand did not stop at परार्ध (10^{12}) but they used to express numbers as

large as (10^{53}). The texts on mathematics, astronomy, geometry like "Shulba Sutras", "Kalpasutras" of Bodhayan, Vaghul, Apastambha, Hiranyakeshin (belonging to Taitariya School) need a special mention. With its 525 sutras in three chapters Bodhyan Shulbasutra is probably the oldest (7th Century B.C.). Both Apastambha and Bodhayan describe a square as the sum of two different squares (like $3^2+4^2=5^2$, $5^2+12^2=13^2$, $8^2+15^2=17^2$, $7^2+24^2=25^2$, $12^2+35^2=37^2$, $15^2+36^2=39^2$.) Katyayan states that if the sides of right angled triangle are a and a $\sqrt{2}$ then the hypotenuse is a $\sqrt{3}$. We can say that the pythagorean theorem has a origin in the shulba sutra.

In the classical period and later we were fortunate in having mathematician. of the stature of Aryabhatt I (5th or 6th century A.D.), Bhaskara 1 (7th century A.D.), Brahmagupta (7th century A.D.), Mahavir (9th century A.D.), Aryabhatt II (10th century A. D.), Shripathy (10th century A.D.), Shridhar (11th Century A.D.), Bhaskara II (12th Century A. D.).

This is an attempt to quote the glorious scientific tradition of our Bharat. This requires presentation of documents & has to be brought to the notice of our students. This book fulfills these requirements. Such efforts are definitely directed to cater to the needs of our upcoming generation of mathematicians. On their behalf I express my heartfelt gratitude to the authors.

- Rambhau Pujari,

G.G. Joshi Shilp Sanshodhan Pratisthan,

Nagpur,

PART (I) - ARITHMETIC AND ALGEBRA

CHAPTER 1

NIKHILAM SUTRA

1.1 Deficiency from the base

The base system is frequently used in Vedic Mathematics to reduce and simplify the computation work involved.

The base is considered to be the powers of ten i.e. $10, 10^2, 10^3$ ....

The multiples of power of ten, i.e. 10, 20, 30, ... 100, 200, 300, ... etc. are also used as bases. These are known as sub-base or working base.

Nikhilam method of multiplication and division always requires a number known as deficiency from base. Observe the following table showing the selection of base and deficiency from it.

The number	The base	The deficiency
110	100	10
1002	1000	2
99	100	- 1
995	1000	- 5
8	10	- 2

Note :- (1) The deficiency is positive if the given number is greater than base and negative if smaller than the base.

(2) The negative deficiency is indicated by bar over the number. e.g. $-5 = \overline{5}$.

Nikhilam Sutra can obtain the deficiency :

Sutra: निखिलम् नवतश्चरमम् दशतः

(Nikhilam navatascaramamdasatah)

(All from nine, and last from ten)

Ex. 1. Find the deficiency of 96 from the base 100.
Using Nikhilam Sutra we get,
last from ten .... (10 - 6) = 4
all from nine .... (9 - 9) = 0,
Hence deficiency = 04.

Ex. 2. Find deficiency of 874 from the base 1000.
last from ten .... (10 - 4) = 6,
all from nine .... (9 - 7) = 2
9 - 8 = 1,
Hence deficiency = 126.

Ex. 3. Find deficiency of 907 from the base 1000.
Deficiency = 9 - 9 | 9 - 0 | 10 - 7 = 093.

Note :- Nikhilam Sutra is not useful in finding the deficiency if the given number is greater than the base.

Exercise: - 1.1

Write the deficiency of following numbers from the corresponding bases.
(1) 978 (2) 9984 (3) 106 (4) 988 (5) 10003.

1.2 Vinculum number:

We have $19 = 20 - 1 = 2\bar{1}$ $\quad$ $27 = 30 - 3 = 3\bar{3}$

$48 = 50 - 2 = 5\bar{2}$ $\quad$ $298 = 300 - 2 = 30\bar{2}$

Observe the new way of representing the numbers. The bar over the digits indicates the negative place value.

e.g. (1) $2\bar{3}4$ Here place value of $\bar{3}$ is (- 30).Hence $2\bar{3}4 = 200 - 30 + 4 = 174.$

(2) $24\bar{1}\bar{3}01$ Here place value of $\bar{1}$ is (- 1000) and that of 3 is (- 300).

Hence $24\bar{1}\bar{3}01 = 200000 + 40000 - 1000 - 300 + 1 = 238701.$

Such numbers in which some digits have negative place values are termed as Vinculum numbers. Any number can be written in Vinculum form thereby avoiding digits greater than 5. Thus if the number contain no digits greater than five the computation becomes so easier and faster. The Vinculum approach is useful for addition, subtraction, multiplication and division and in developing multiplication tables.

1.3 Use of Nikhilam Sutra in writing number in Vinculum form.

Ex. 4. Write 238 in Vinculum form.

Steps 1. Mark the digits greater than 5. Here it is 8.

2. Use Nikhilam Sutra for the digit 8.

We have 10 - 8 = 2

and increase 3 by one. Thus $238 = 24\bar{2}$.

Ex. 5. Write 3297016783 in Vinculum form.

Steps 1. The groups of digits, which are to be converted in

vinculum form, are 1 (6, 7, 8) & 2 (9, 7).

2. From first group using Sutra we get :

10 - 8 = 2, 9 - 7 = 2, 9 - 6 = 3, 1 + 1 = 2

From second group : 10 - 7 = 3, 9 - 9 = 0, 2 + 1 = 3

3. Thus 3 2 9 7 0 1 6 7 8 3 = $3\ 3\ 0\ \bar{3}\ 0\ 2\ \bar{3}\ \bar{2}\ \bar{2}\ 3$.

1.*4* Conversion of negative number into vinculum form

Ex. 6. Convert - 38 into Vinculum form.

Steps 1) We write - 38 = $0\overline{38}$

2) By Nikhilam Sutra we get,

$0\overline{38}$ = 0 - 1 | 9 - 3 | 10 - 8 | = $\bar{1}$ 6 2.

Ex. 7. Convert - 5696 into Vinculum form.

We write - 5 6 9 6 = $0\ \bar{5}\ \bar{6}\ \bar{9}\ \bar{6}$ = $\bar{1}$ 4 3 0 4.

1.5 Conversion of numbers in normal form.

Ex. 8. Convert 2 $4\bar{2}$ into normal form.

Steps 1) Mark the digits which have negative place value i.e. bar over-head.

2) Use Nikhilam Sutra and decrease 1 from 4.

Thus 2 $4\bar{2}$ = 2 | 4 - 1 | 10 - 2 | = 2 3 8.

Ex. 9. For 9 $\overline{8903}$ = 9 | 8 - 1| 9 - 9 | 9 - 0 |10 - 3| = 97097

Ex.10. For $1\overline{38}$ = 1-1 | 9-3 | 10 - 8 | = 0 6 2 = 62

1.6 Multiplication Table and Vinculum

Ex.11. Prepare a multiplication table for 19.

Steps ***1)*** Write 19 in Vinculum form : i.e. $19 = 2\bar{1}$

2) The operator at unit's place is $\bar{1}$ and digit at unit's place is 9. Go on reducing this digit by 1.

3) The operator at ten's place is 2 and digit at this place is 1. Go on increasing this digit by 2.

Thus we get a table for 19.

$2\bar{1}$

1 9
3 8
57
76
9 5
1 1 4
1 3 3
1 5 2
1 7 1
1 9 0.

Ex.12. Prepare a multiplication table for 87.

Steps [1] Here $0\ 8\ 7 = 1\bar{1}\bar{3}$

[2] The operator at unit's place is $\bar{3}$, at ten's place $\bar{1}$ and hundred's place 1. Thus multiplication table is :

	1	$\bar{1}$	$\bar{3}$	
	0	8	7	
	1	7	4	
	2	6	1	
	(3	5	$\bar{2}$)	Converting to normal form
=	3	4	8	
	4	3	5	
	5	2	2	
	6	0	9	
	6	9	6	
	7	8	3	
	8	7	0.	

Ex.13. Prepare a multiplication table for 8 9 7.

Steps 1) $897 = 1\bar{1}0\bar{3}$.

2) The operator at unit's place $\bar{3}$, at ten's place 0, at hundred's place $\bar{1}$ and at thousand's place 1. Thus multiplication table is :

1	$\bar{1}$	0	$\bar{3}$
0	8	9	7
1	7	9	4
2	6	9	1
(3	5	9	$\bar{2}$)
= 3	5	8	8
4	4	8	5
5	3	8	2
6	2	7	9
7	1	7	6
8	0	7	3
8	9	7	0.

1.7 Use of Vinculum in subtraction

Ex.14.

$$\begin{array}{r} 4\ 7\ 8 \\ -\ 3\ 2\ 5 \\ \hline 1\ 5\ 3 \end{array}$$

If upper digit is greater, then write down the difference directly.

Ex.15.

$$\begin{array}{r} 3\ 2\ 4 \\ -\ 2\ 7\ 6 \\ \hline 1\ \bar{5}\ \bar{2} \end{array} \quad = 0\,4\,8 = 48$$

Ex.16.

$$\begin{array}{r} 5\ 6\ 3\ 8\ 1 \\ -1\ 9\ 6\ 7\ 0 \\ \hline 4\ \bar{3}\ \bar{3}\ 1\ 1 \end{array} \quad = \ 3\ 6\ 7\ 1\ 1$$

Note :- By writing the subtraction in Vinculum form we avoid
" borrowing the digit " part.

1.8 ADDITION

Ex.17. Find 8985 + 2376 + 4889 + 3605 + 7512.

Note: As per the present method of the addition we begin to add digits from unit's place downwards or upwards. We write the carry digit (if any) over-head and then perform addition at ten's place etc.

But in Vedic method as the sum exceeds nine, increase the digit in left column by one by marking a star (*) over-head.

The symbols used are

$8^* = 8 + 1 = 9$, $0^* = 0 + 1 = 1$, $9^* = 9 + 1 = 0$, carry 1

Answer:

	8	9	8	5
0*	2*	3*	7*	6
	4*	8*	8*	9
	3	6	0	5
0*	7*	5	1	2
2	7	3	6	7

Exercise:- 1.2

[A] Write the following numbers in Vinculum form.

1) 2 8 3 7. 2) 9 8. 3) 1 2 9.

[B] Write the following numbers in normal form (Shudha Rupa)

1) 321 0 4 2) 1 $\bar{1}\,\bar{1}\,\bar{1}$. 3) 3 $\bar{2}$ $\bar{2}$ $\bar{2}$ $\bar{2}$.

[C] Prepare multiplication table for.

1) 8 9 2) 1 3 9

[D] Convert the following negative numbers into Vinculum form.

1) - 3 8 2) - 1 2 5. 3) - 1 2 3 6

[E] Find the value of:

1) 7 3 2 1 9 - 1 8 5 3 2.

2) 3 2 8 + 1 2 9 + 4 7 0 + 1 2 5 + 1 0 0.

3) 1 2 3 6 7 + 5 4 3 0 - 8 2 4 9

Answers:1.2

[A] (1) 3 $\bar{2}$ 4 $\bar{3}$ (2) 1 0$\bar{2}$ (3) 1 3$\bar{1}$

[B] (1) 27904 (2) 889 (3) 2778

[D] (1) $\bar{1}$ 6 2 (2) $\bar{2}$ 7 5.

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CHAPTER 2.

MULTIPLICATION

2.1 Our present day `aprocedure of multiplication requires multiplication tables. But according to Vedic System the multiplication table above 5 are not required. Urdhava-Tiryak Sutra is the most general formula, which can be applied to all cases. But Nikhilam Sutra and its Upa Sutra can be applied only to special cases. First we discuss these special cases.

2.2 Nikhilam Method:

Sutra: निखिलम् नवतश्चरम् दशतः Nikhilam navatascaramamdasatah

(All from nine, and last from ten)

Case *(I)* When both the numbers are smaller than base.

Ex.1. Find 9 x 8.

Steps *(1)* Select a base, which is nearest to given numbers. Here base = 10.

(2) Subtract each of the numbers to be multiplied from base.

Here 10 - 9 = 1, 10 - 8 = 2. These deficiency numbers are written as shown.

$$\begin{array}{lcc} 9 & : & \overline{1} \\ 8 & : & \overline{2} \\ \hline 7 & : & 2 \end{array}$$

(3) Here numbers to be multiplied are smaller than the base hence these deficiency numbers are treated as negative numbers or bar digits.

(4) We get the answer (product) in two parts.

1) Right-hand part of the answer is $\overline{1} \times \overline{2} = 2$.

The number of digits in the right-hand part should be equal to n where 10^n is base number.

(5) The cross addition gives left-hand part of the answer.

$9 + \overline{2} = 7$ or $8 + \overline{1} = 7$

Ex.2. ***Find 97*** x ***88.*** ***Steps***

97	:	$\overline{0}\ \overline{3}$	1) Base = 100 = 10^2.
88	:	$\overline{1}\ \overline{2}$	2) 100 - 97 = 03, 100 - 88 = 12.
85	:	36	3) $\overline{03} \times \overline{12} = 36$
Ans :	8536		4) 97 + 12 = 85 or 88 +03 = 85.

Ex.3. ***Find 999*** **X** ***892.*** ***Steps***

999	:	$\overline{0}\ \overline{0}\ \overline{1}$	1) Base = 1000 = 10^3.
892	:	$\overline{1}\ \overline{0}\ \overline{8}$	2) 1000 - 999 = 001,
891	:	1 0 8	1000 - 892 = 108.
891	:	1 0 8	3) $\overline{001} \times \overline{108} = 108$
Ans :	891108		4) $892 + \overline{0}\ \overline{0}\ \overline{1} = 891$

Ex.4. ***Find 88*** x ***85.*** ***Steps***

88	:	$\overline{1}\ \overline{2}$	1) Base = 100 = 10^2
85	:	$\overline{1}\ \overline{5}$	2) 100 - 88 = 12, 100 - 85 = 15
73	:	180	3) $\overline{1}\ \overline{2} \times \overline{1}\ \overline{5} = 180$
74	:	80	Number of digits in the right-hand part of the answer should be 2.
Ans	7480		

The surplus digit (1) is carried over higher level to write 73 + 1 = 74.

Case (II) When both numbers to be multiplied are greater than the base.

Ex.5. ***Find 1 1 2*** x ***1 0 9*** ***Steps***

112	:	12	1) Base = 100 = 10^2.
109	:	09	2) 112 - 100 = 12, 109 - 100 = 09.
121	:	108	3) 12 x 09 = 108, 112 + 09 = 121
122	:	08	4) 121 + 1 = 122
Ans	12208.		

Ex.6. ***Find 1 0 0 1*** X ***1 0 0 2*** ***Steps***

1001	:	001	1) Base = 1000 = 10^3.
1002	:	002	2) 1001 - 1000 = 1,
1003	:	002	1002 - 1000 = 2.

3) 1 x 2 = 2, 1001 + 002 = 1003.

Ans 1003002.

Right-hand part should contain 3 digits as the base is 100 hence we insert two zeros in this part.

Case (III) when one number is greater than the base and other is smaller one.

Ex.7. ***Find 1 1 5 x 9 8*** ***Steps***

115	:	1 5	1) Base = 100 = 10^2
98	:	$\bar{0}\ \bar{2}$	2) 115 - 100 = 15, 98 - 100 = $\bar{0}\ \bar{2}$
113	:	30	3) 15 x $\bar{2}$ = $\bar{3}\ \bar{0}$
			98 + 15 = 113 = 115 - 2

Ans 11300 + $\bar{3}\ \bar{0}$ = 11270

Ex.8. ***Find 1 0 2 7 x 9 9 8*** ***Steps***

1027	:	0 2 7	1) Base = 1000 = 10^3
998	:	$\bar{0}\ \bar{0}\ \bar{2}$	2) 1027 - 1000 = 027
			998 - 1000 = $\bar{0}\ \bar{0}\ \bar{2}$
1025	:	$\bar{0}\ \bar{5}\ \bar{4}$	3) 27 x $\bar{2}$ = $\overline{54}$, 1027 - 2 = 1025

Ans 1 0 2 5 $\bar{0}\ \bar{5}\ \bar{4}$ = 1024946

Ex.9. ***Find 1 1 5 x 8 8*** ***Steps***

115	:	1 5	1) Base = 100 = 10^2
88	:	$\bar{1}\ \bar{2}$	2) 115 - 100 = 15, 88 - 100 = $\overline{12}$
103	:	$\overline{1 8 0}$	3) 15 x $\overline{12}$ = $\overline{180}$, 115 - 12 = 103

Ans (103 - 1) : $\overline{80}$ or 88 + 15 = 103.

= $102\overline{80}$ = 10120. 4) Surplus (- 1) is carried to left hand part.

Ex.10 ***Find 97*** x ***1001*** ***Steps***

$$\begin{array}{lcl} 097 & : & \bar{9}\,\bar{0}\,\bar{3} \\ 1001 & : & 0\,0\,1 \\ \hline 98 & : & \bar{9}\,\bar{0}\,\bar{3} \end{array}$$

1) Base = 1000 = 10^3

2) 97 - 1000 = $\overline{903}$,

1001 - 1000 = 1

3) $\overline{903} \times 1 = \bar{9}\ \bar{0}\ \bar{3}$, 97 + 1 = 98

Ans $98\,\bar{9}\,\bar{0}\,\bar{3} = 97097$ or 1001 - 903 = 98.

2.3 Algebraic explanation of Nikhilam method

Let the numbers to be multiplied be (x + a) and (x + b) where x is the base, and a , b are deficiencies from the base.

Consider the product :

$$(x+a)(x+b) = x^2 + xa + xb + ab$$

$$= x(x+a+b) + ab.$$

Now by Nikhilum method we get

$$\begin{array}{lcl} x+a & : & a \\ x+b & : & b \\ \hline \end{array}$$

$$= (x+a+b) : ab$$

According to the place value Ans = $x(x+a+b) + ab$.

Note: The digit 'a b' is at unit place where as digit (x + a + b) is at next higher place.

2.4 Nikhilam and Anurupyena method

Sutra : आनुरूप्येण (anurupyena)

(Proportionately)

Ex.11. Find 53* x *55

First we have to select a proper base which is nearer to both 54 and 55. We select the base 100. The deficiencies are 47 and 45 respectively and thus the problem is not simplified by choice of 100. So some other base should be selected. The other bases are 10, 20, 30 etc. These bases are known as working bases or sub-bases. The Upa-Sutra Anurupyena helps in finding answer using working base.

	53	: 3	1)	Theoretical base = 100,
	55	: 5		Working base = 50 = 100 / 2.
	————			
	(1 / 2). 58	: 15	2)	53 - 50 = 3, 55 - 50 = 5.
i.e.	29	: 15	3)	3 5 = 15.
Ans	2915.		4)	(55 + 3) / 2 = 28 = (53 + 5) / 2.

Note :- 1) Ratio of left-hand part of answers is (29/58)=1/2.

2) The ratio R = (working base / theoretical base)

= 50 / 100 = 1 / 2.

3) Hence the Sutra is known as Anurupyena

(Proportionately)

Ex.12. Find 72* x *78 — ***Steps***

72	:	2	1) Theoretical base = 10,
78	:	8	2) Working base = 70
7 (x 80)	:	16	3) 72 - 70 = 2, 78 - 70 = 8,
560	:	16	4) 2 x 8 = 16
Ans 5616			5) Surplus (1) is carried to left - hand part.

Ex.13. Find 43* x *45 — ***Steps***

43	:	07	1) Theoretical base = 100,
45	:	05	2) Working base = 50
1 / 2(38)	:	35	3) 43 - 50 = 07, 45 - 50 = 05,
19	:	35	4) 7 x 5 = 35.
Ans	1935		5) Left-hand part of the answer = (45 - 7) / 2 = (43 - 5) / 2 = 38 / 2 = 19.

2.5. *Algebraic explanation*

Let the theoretical base be x, working base be y and their ratio be R. so that y =Rx. Let a & b be the deficiencies from working base y. Hence the numbers are (y + a) and (y + b). Consider the product (y + a) . (y + b).

$$(y+a).(y+b) = (Rx+a).(Rx+b).$$
$$= R^2x^2 + Rxb + Rxa + ab.$$
$$= Rx(Rx+a+b) + ab.$$
$$= Rx(y+a+b) + ab.$$

Thus right-hand part of the answer is ab & left-hand part of the answer is $Rx(y+a+b)$.

This algebraic explanation indicates :

1). The right-hand part of the answer remains unchanged.

2). The left-hand part of the answer is to be made proportionate (Anurupa) by multiplying by R.

2.6 *Ekanyunena Purvena Method.*

Sutra: एकन्यूनेन पूर्वेण: ekanunen Purvena

(One less than the previous)

This Upa-Sutra is used in the special case when the multiplier consists of all nines.

Case 1. When multiplier contains equal or greater number of nines.

Ex.14. Find 481 x 999

Steps (1) By above Upa-Sutra left-hand part of the answer is 481 - 1 = 480.

(2) Right-hand part of the answer is compliment of nine of each digit in left-hand part.

Thus 9 - 4 = 5, 9 - 8 = 1, 9 - 0 = 9

Hence left-hand part is 519.

(3) Answer = 480519.

Ex.15. Find 578* x *9999.

We write 578 x 9999

= 0578 x 9999

= 0577 | (9-0) | (9-5) | (9-7) | (9-8)

= 0577 | 9422

= 5779422 Answer.

Case 2. When multiplier contains less number of nines.

Ex.16. Find 327* x *99.

Steps (1) By above Upa-Sutra we get 327 - 1 = 326.

(2) Put multiplier 99 to its right side. We get 32699.

(3) Subtract 326 from 32699.

We get 32699 - 326 = 32373.

(4) Answer is 32373.

2.7. Urdhva Tiryak Method

Sutra: उर्ध्व तिर्यग्भ्याम् Urdhva tiryakbhyam

(Vertically and crosswise)

This Sutra is most general formula applicable to all cases of multiplication.

Ex.17. Find 21* x *14.

Steps (1) For unit's place: 1x 4 = 4

(2) For ten's place: (2 x 4) + (1 x 1) = 9.

(3) For hundred's place: 2 x 1 = 2.

(4) Answer = 294.

Ex.18. Find 532* x *472.

Steps (1) For unit's place : 2 x 2 = 4

(2) For ten's place : (3 x 2) + (7 x 2) = 20

(3) For hundred's place : (5 x 2) +(4 x 2)+(7 x 3) =39

(4) For thousand's place : (7 x 5) + (4 x 3) = 47

(5) For ten thousand's place : (5 x 4) = 20

(6) The answer is

$$\begin{array}{r} 0\,7\,9\,0\,4 \\ +\ 2\,4\,3\,2\,0 \\ \hline 2\,5\,1\,1\,0\,4 \end{array}$$

The above steps can be written in short as follows :

$$\begin{array}{rccc} & 5 & 3 & 2 \\ \text{x} & 4 & 7 & 2 \\ \hline \end{array}$$

20 | 47 | 39 | 20 | 04

= 2 5 1 1 0 4 (Adding part wise)

Ex.19. Find 1021 x 2103

$$
\begin{array}{ccccc}
 & 1 & 0 & 2 & 1 \\
\times & 2 & 1 & 0 & 3 \\
\hline
\end{array}
$$

$$2\,1\,4\,7\,1\,6\,3$$

Steps

[1] 1 x 3 = 3

[2] (3 x 2) + (1 x 0) = 6

[3] (3 x 0) + (1 x 1) + (2 x 0) = 1

[4] (3 x 1) + (1 x 2) + (1 x 2) + (0 x 0) = 7

[5] (1 x 0) + (2 x 2) + (1 x 0) = 4

[6] (2 x 0) + (1 x 1) = 1

[7] (1 x 2) = 2

Ex.20. Multiply 889 x 898.

$$
\begin{array}{ccccc}
 & 1 & \bar{1} & \bar{1} & \bar{1} \\
\times & 1 & \bar{1} & \bar{0} & \bar{2}
\end{array}
$$

$$1\ \bar{2}\ \bar{0}\ \bar{2}\ 3\ 2\ 2$$

$$=\ 7\,9\,8\,3\,2\,2$$

Note : $889 = 1\,\bar{1}\,\bar{1}\,\bar{1}$ and $898 = 1\,\bar{1}\,\bar{0}\,\bar{2}$

2.8. Multiplication And Addition / Subtraction :

Multiplication and addition can be performed simultaneously with the help of Urdhva Tiryak Sutra as explained by following examples.

Ex.21. Find (23 x 47) + (36 x 19) - (61 x 12).

Answer : We write the structure as follows.

23		36		61
	+		-	
47		19		12

(8 + 3 - 6) | (14 + 12 + 27 + 6 - 12 - 1) | (21 + 54 - 2)

= 5 | 46 | 73

= 1 0 3 3

Steps [1] (2 x 4) + (3 x 1) - (6 x 1) = 5

[2] (2 x 7)+(4 x 3)+(3 x 9)+(6 x 1)-(6 x 2)-(1 x 1) = 46

[3] (3 x 7) + (6 x 9) - (1 x 2) = 73

Ex.22. Find (145 x 273) - (301 x 52)

Ans we write the structure as follows.

	145		301
		-	
x	273		052

(2 - 0) | (7 + 8 - 15)|(3 + 10 + 28 - 6)|(12 + 35 - 5)|(15 - 2)

= 2 | 00 | 35 | 42 | 13

= 2 3 9 3 3.

Ex.23. Find (2431 x 36) + (342 x 1723).

Ans we write the structure as follows.

$$\begin{array}{cccc} & 2431 & & 0342 \\ & & + & \\ \times & 0036 & & 1723 \\ \hline \end{array}$$

$$= \quad 3 \mid 31 \mid 60 \mid 64 \mid 37 \mid 12$$

$$= \quad 6\,7\,6\,7\,8\,2.$$

Exercise: 2.1

[A] Find the product by using Nikhilam method.

[1] 98 x 94 [2] 993 x 894 [3] 105 x 109

[4] 1003 x 1007 [5] 102 x 97 [6] 1014 x 989.

[B] Find the product by using Anurupyena Method

[1] 68 x 62 [2] 39 x 43 [3] 71 x 73

[4] 213 x 204.

[C] Find the product by using Ekanyunena method.

[1] 56 x 99 [2] 4050 x 9999 [3] 315 x 999

[4] 999 x 47 [5] 3248 x 9999 [6] 272 x 9

[7] 512 x 99 [8] 8125 x 999

[D] Find the product using Urdhva Tiryak method.

[1] 54 x 21 [2] 103 x 214 [3] 65 x 37

[4] 4531 x 2361 [5] 979 x 212

[6] 1004 x 3273 [7] (23 x 32) + (43 x 51)

[8] (120 x 14) - (52 x 23)

[9] (24 x 28) + (52 x 39) - (61 x 15)

[10] (224 x 58) + (327 x 512) - (1031 x 53)

Exercise: - 2.2

Find the product by using suitable method

[1] 8 12 [2] 98 105 [3] 999 990

[4] 100073 9999 [5] 46 44 [6] 999 249

[7] 315 1005 [8] 378 999 [9] 32142 1232

[10] 12543 124

Answers 2.1

[A] [1] 9212 [2] 887742 [3] 11445 [4] 1010021 [5] 9894
[6] 1002846.

[B] [1] 4216 [2] 1677 [3] 5183 [4] 43452

[C] [1] 5544 [2] 40495950 [3] 314685 [4] 46953
[5] 32476752[6] 2448 [7] 50688 [8] 8116875

[D] [1] 1134 [2] 22042 [3] 2405 [4] 10607691 [5] 207548
[6] 3286092 [7] 2929 [8] 3413 [9] 1785 [10] 125826

Answers 2.2

[1] 96 [2] 10290 [3] 989010

[4] 100072899927 [5]2024 [6] 248751

[7] 316575 [8] 377622 [9] 39598944

[10] 1555332

❖ ❖ ❖

CHAPTER 3.

SQUARES AND SQUARE ROOTS

3.1 The method of finding the square and square roots of the number is based on duplex of the number so we first study the evaluation of duplex of the number.

SUTRA : द्वंद्वयोग: (dvandvayogah) (Duplex)

The term Duplex means

1. For single digit number a , $D(a) = a^2$

2. For double-digit number a b , $D(ab) = 2ab$.

3. For three digit number a b c , $D(abc) = 2ac + b^2$.

4. For four digit number a b c d ,

$D(abcd) = 2ad + 2bc$. etc

Illustrations:—

1. $D(3) = 3^2$

2. $D(35) = (2)(3)(5) = 30$.

3. $D(438) = D(48) + D(3) = (2)(4)(8) + 3^2 = 73$.

4. $D(1437) = D(17) + D(43) =$

$(2)(1)(7) + (2)(4)(3) = 38$

5. D (93241) = D (91) + D (34) + 2^2 .= 18 + 24 + 4 = 46.

3.2 *Square of the number*

To find the square of the number, we add the duplex of digits therein successively, part wise.

Illustrations:—

[1] 3^2 = D (3) = 9.

[2] 35^2 = D (3) | D (35) | D (5) = 9 | 30 | 25 = 1225.

[3] 47.1^2 = D (4) | D (47) | D (471) | D (71) | D (1)

= 16 | 56 | 57 | 14 | 01 | = 2218 .41.

(We ignore for a while the decimal point to calculate the Duplex).

[4] 5032^2 = 25 | 00 | 30 | 20 | 09 | 12 | 04 | = 25321024.

3.3 *Checking Method*

We check the answer by Beejank method.

In Ex. 2, Beejank of 35=B(35)=3+5= 8 and B (8^2) = 1 similarly

B (1225) = 1.As the Beejankas are same the answer may be correct.

3.4 *Special Cases.*

Case 1 :- Sutra : यावदूनम् तावदूनीकृत्य वर्गं च योजयेत्

Lessen the deficiency from the base (selected) to find the square.

Ex.1. Find the square of 94.

[1] We select the base as 100.

[2] $100 - 94 = 6$.

[3] Hence $94 - 6 = 88$ is right hand part of the answer.

[4] $(6)^2 = 36$ is the left-hand part of the answer.

Thus $94^2 = (94 - 6) \mid (6)^2 = 88 \mid 36 = 8836$.

Ex. 2:- $87^2 = (87 - 13) \mid (13^2)$

$= 74 \mid 169 = 7569$ { Part wise addition }.

Ex. 3:- $103^2 = (103 - (- 3)) \mid (- 3)^2 = 106 \mid 09 = 10609$.

Algebraic Explanation :- Let given number be 'a' and nearest base is 'b'. Let deficiency from the base is x (x may be +ve or - ve).

Now $a = b + x$ gives

$a^2 = (b + x)^2 = b^2 + 2bx + x^2 = b (b + x + x) + x^2$.

Hence $a^2 = b (a + x) + x^2$.

Case 2 :- Sutra - एकाधिकेन पूर्वेण

This method is applicable to find the squares of the numbers ending in 5 only

Ex. 1:- $(65)^2 = 6 (6 + 1) \mid 5^2 = 42 \mid 25 = 4225$.

Ex. 2:- $(105)^2 = 10 (10 + 1) \mid 5^2 = 110 \mid 25 = 11025$.

Algebraic Explanation :- Let given number be $10a + 5$.

Then $(10a + 5)^2 = 100a^2 + 100a + 5^2 = 100a (a + 1) + 25$.

EXERCISE: 3.1

[A] Find the squares of the following numbers by Duplex Method and check the answers.

[1] 47 [2] 324 [3] 24.03 [4] 7813

[5] 61058 [6] 98997 [7] 123456 [8] 68.31

[9] 212.4 [10] 889.9

[B] Find the square by Yavadunum Method.

[1] 98 [2] 101 [3] 89 [4] 1002

[5] 998 [6] 100003 [7] 9988

[C] Find the square by Ekadhikena Method:

[1] 85 [2] 205 [3] 55 [4] 1005

[5] 995.

ANSWERS 3.1

[A] [1] 2209 [2] 104976 [3] 5774409 [4] 61042969

[5] 37280793646 [6] 98004060097

[7] 15241383936 [8] 4666.2561 [9] 45113.76

[10] 791922.01

[B] [1] 9604 [2] 10201 [3] 7921 [4] 1004004

[5] 996004 [6] 10000600009 [7] 99760144.

[C] [1] 7225 [2] 42025 [3] 3025 [4] 1010025

[5] 990025.

3.5 Sums And Differences Of Squares.

We have already studied the method of finding squares of numbers by

' Duplex Method '. Same duplexes are useful in finding sums and differences of squares of numbers when arranged in proper structure.

Case 1:

Sums of the squares.

Method :- Add place wise the duplexes of digits of numbers, whose squares are to be added or subtracted. The method is explained by following examples

Ex. [1] Find $24^2 + 13^2$.

Ans. We write duplexes for 24^2 and 13^2 place wise as shown.

$$
\begin{array}{lcccc}
24^2 + 13^2 & = & 4 & 16 & 16 \\
 & + & 1 & 6 & 9 \\
\hline
 & & 5 & 22 & 25 \\
\hline
\end{array}
$$

Now We add sum of the duplexes 5| 22 | 25 part wise, hence

$$24^2 + 13^2 = 7\,4\,5\,.$$

Ex.[2] Find $89^2 + 68^2$.

By Vinculum we write $89 = 9\bar{1}$, $68 = 7\bar{2}$.

Now $(9\bar{1})^2 + (7\bar{2})^2$

$$
\begin{array}{rccc}
= & 81 & \bar{1}\bar{8} & 01 \\
+ & 49 & \bar{2}\bar{8} & 04 \\
\hline
 & 130 & \bar{4}\bar{6} & 05 \\
\hline
\end{array}
$$

Adding part wise we get: $1\,3\,\bar{4}\,\bar{6}\,5 = 1\,2\,5\,4\,5$.

Hence $89^2 + 68^2 = 1\,2\,5\,4\,5$

Ex.[3] Find $232^2 + 430^2$.

$$
\begin{array}{rrrrrrr}
232^2 + 430^2 & = & 04 & 12 & 17 & 12 & 04 \\
 & + & 16 & 24 & 09 & 00 & 00 \\
\hline
 & & 20 & 36 & 26 & 12 & 04 \\
\hline
\end{array}
$$

Adding part wise we get $232^2 + 430^2 = 2\,3\,8\,7\,2\,4$

Case 2: Difference of squares

Ex.[4] Find $73^2 - 41^2$.

$$
\begin{array}{rrrrr}
73^2 - 41^2 & = & 49 & 42 & 09 \\
 & - & 16 & 08 & 01 \\
\hline
 & & 33 & 34 & 08 \\
\hline
\end{array}
$$

Adding part wise we get : 33 | 34 | 08 = 3648

Hence $73^2 - 41^2 = 3648$.

Ex.[5] Find $5412^2 - 3016^2$.

$$
\begin{array}{rccccccc}
5412^2 - 3016^2 = & 25 & 40 & 26 & 28 & 17 & 04 & 04 \\
- & 09 & 00 & 06 & 36 & 01 & 12 & 36 \\
\hline
 & 16 & 40 & \overline{20} & \overline{12} & 16 & \overline{12} & \overline{3}\,\overline{2} \\
\hline
\end{array}
$$

Adding part wise we get: $2\ 0\ 2\ \overline{1}\ 3\ 5\ \overline{1}\ \overline{2}$

Hence $5412^2 - 3016^2 = 20193488$.

Exercise: 3.2

Find the value of :-

[1] $32^2 + 71^2$ [2] $42^2 + 63^2$ [3] $103^2 + 210^2 + 314^2$

[4] $23^2 + 116^2 + 89^2$

[5] $2305^2 + 4108^2 + 371^2$ [6] $78^2 - 35^2$

[7] $405^2 + 58^2 - 216^2$ [8] $61^2 - 43^2$

[9] $1304^2 + 261^2 - 719^2$

[10] $315^2 - 712^2 + 531^2$

Answers: 3.2

[1] 6065 [2] 5733 [3] 153305

[4] 21906 [5] 22326330 [6] 4859

[7] 120733 [8] 1872 [9] 1251576

[10] - 125758

3.6 SQUARE ROOTS

We explain the method of finding the square roots by examples.

Ex.1:- Find the square root of 552049.

The table of operations is as follows.

Remainder line		6 6 2 0 0
Given number		$5\ 5 \mid 2\ 0\ 4\ 9$
Divisor	14	
Answer line		$7 \mid 4\ 3\ 0\ 0$

Step1. We group the digits of the given number in pairs from right to left as $55 \mid 20 \mid 49$.

Note : The left most group may contain one digit.

Step 2. Find the number whose square is nearest to left most group.

Here $7^2 = 49$ is nearest to 55, hence 7 is the first digit in the answer line, 55 - 49 = 6 is the first digit in remainder line and (2 x 7) = 14 is divisor.

Step 3. Considering remainder 6 and next digit of given number i.e. 2, we get the dividend 62.This 62 / 14 gives Quotient Q = 4 and

Remainder R = 6.

Step 4 Now 60 - D (4) = 60 - 16 = 44. And 44 / 14 gives Q= 3 and R = 2.

Step 5. 24 - D (43) = 24 - 24 = 0. And 0 / 14 gives Q = 0 and R = 0.

Step 6. 09 - D (43 0) = 9 - 9 = 0. And 0 / 14 gives Q = 0 and R = 0.

Step 7. There are three groups in the given number, hence decimal point in the answer is placed after three digits counted from left.

Answer : The square root of 552049 is 743 .00

Ex. 2. Find the square root of 2142.

The table of operations is as follows.

Remainder line		5 6 10 12 12
Given number		2 1 | 4 2 0 0 0
Divisor	8	
Answer line		4 | 6 2 8 1

Step 1. Group the digits as 21 | 42. Now $4^2 = 16$ is nearest to 21

Hence 4 is the is first digit in answer line, first remainder is 5 and divisor is 8.

Step 2. 54 / 8 gives Q = 6 and R = 6.

Step 3. 62-D(6)=62-36=26, and 26/8 gives Q=2 and R=10.

(Note the choice of Q and R to avoid negative dividend.)

Step 4. 100 - D (62) = 100 - 24 = 76, and 76 / 8 gives Q = 8 and R = 12.

Step 5. 120 - D (628) = 120 - 100 = 20, and 20 / 8 gives Q = 1 and R = 12.

(The procedure can be continued up to desired number of decimal places.)

Answer :- The square root of 2142 is 46. 281.....

Ex. 3 :- Find the square root of 0. 16384.

The table of operations is as follows.

Remainder line		0 3 6 8
Given number		0 | 1 6 | 3 8 4 0
Divisor	8	
Answer line		0 | 4 0 4 7 7

Step 1. When integer part is zero, we group the decimalpart from left to write as 0. | 16 | 38 | 40 .... Now 4 = 16 is nearest to 16, hence 16 - 16 = 0 is first divisor. Remaining procedure is same.

Answer :- The square root of 0.1638 is 0.4077

Ex. 4:- Find the square root of 17956.234

The table of operations is as follows.

Remainder line		0 1 2 1 0 0
Given number		1 | 7 9 5 6 2 3 4
Divisor	2	
Answer line		1 | 3 4 0 0 0 8 ---

Note:- Group the integral part of given integer as 1 | 79 | 56 .Remaining procedure is same as above.

Answer:- The square root of 17956. 234 is 139. 0008——

3.7 Square root (Special Case).

SUTRA : **विलोकनम्** (vilokanam) (By Observation)

The square root of the number, which is perfect square,

and comparatively smaller one, can be well judged by observations

and then can be confirmed by Beejank method . We study this method.

Observations **:-**

[1] The square of the number, with Right Most Digit (RMD) 1 or 9 has RMD 1. Conversely the square root of the number, with RMD 1, has 1or 9.

[2] Similarly square root of the number with RMD 4 has 2 or 8 as RMD.

[3] Square root of the number with RMD 5, has 5 as RMD, with RMD 6 has 4 or 6 as RMD, and with RMD 9 has 3 or 7 as RMD.

[4] No perfect square number has 2, 3, 7, and 8 as RMD.

Method :- ***To find square root of given number***

Steps:

[1] We break the given number into two parts such that right most part

(RMP) should contain only two digits.

[2] By observing right most digit of RMP, fix the likely right most digits

of the square root with refer to observations above.

[3] Fix the number whose square is nearest less to the left most part (LMP).

[4] Guess the likely square root from Step 2 and 3.

[5] Confirm the answer by Beejank method.

Illustrative examples

Ex.1:- Find square root of 8281.

Steps:-

[1] We divide the given number into two parts (82 | 81)

[2] As right most digit in RMP is 1, right most digit in the square root is either 1 or 9.

[3] 9 is the number whose square 81 is nearest less to LMP

[4] Hence square root of (8281) is either 91 or 99.

[5] Now $B(8281) = 1$,

[6] $B\{[B(91)]^2\} = B\{[1]^2\} = B\{1\} = 1$,

[7] $B\{[B(99)]^2\} = B\{[9]^2\} = B\{81\} = 9$,

[8] From Step 4 and 6 we confirm that square root of given number is 91

Ans :— **Square root of 8281 is 91.**

Exercise

[A] Find <u>Square roots</u> of the following numbers and <u>check</u> the answers

[1] 1936 [2] 20736 [3] 2199289

[4] 9641025 [5] 543 [6] 14535

[7] 547.56 [8] 78.5938 [9] 0.016384

[10] 0.00996.

[B] Find the square root by observation only

[1] 784 [2] 10609 [3] 5776

[4] 18225

Answers

[A]	[1] 44	[2] 144	[3] 1483
	[4] 3105	[5] 23.3023	[6] 120.561
	[7] 23.397	[8] 8.8653	[9] 0.128
	[10] 0.0997.		
[B]	[1] 28	[2] 103	[3] 76
	[4] 135		

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CHAPTER 4.

CUBES AND CUBE ROOTS

4.1 In this chapter, we shall study how to evaluate cube of two digit number, (only) with the algebraic formula $(a+b)^3$. The use of Vinculum numbers ease the calculations.

4.2 Method

1) We have $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

2) For any two digit number x y,

   we assume x as digit at ten's place and y as digit at unit place.

3) We write the structure as follows.

$(xy)^3$ =	x^3	x^2y	xy^2	y^3
+		$2x^2y$	$2xy^2$	
	x^3	$3x^2y$	$3xy^2$	y^3

4) We add column wise first and then part wise. For two-digit number each part will contain one digit

Note :-

Number	1	2	3	4	5	6	7	8	9
Cube	1	8	27	64	125	216	343	512	729

Ex. 1) Find $(14)^3$.

Ans. : We write the structure as follows.

$14^3 = 01$	04	16	64
+	08	32	
= 01	12	48	64

= 01 | 2 | 8 | 4 Base = 10 hence each part contains

= 01 | 4 | 6 | one digit. Remaining are carried.

= 2 7 4 4.

Ex. 2) Find $(92)^3$.

Ans. : We write the structure as.

92^3	=	729	162	36	08
	+		324	72	
	=	729	486	108	08
	=	729	6	8	8
		48	0	0	
		1			
	=	778	6	8	8

by adding part wise.

Ex. 3) Find (55)^3.

Ans. : We write the structure as.

55^3	=	1 2 5	1 2 5	1 2 5	1 2 5
	+		2 5 0	2 5 0	
	=	1 2 5	3 7 5	3 7 5	1 2 5
	=	1 6 6 3 7 5		*by adding part wise.*	

4.3 Special case :-

Sutra:- यावदूनम् Yavadunam

(whichever is deficient)

When the numerator is nearer to the base 10,100,1000 etc the cube can be evaluated by upa-sutra "Yavaddunam".

Let (x + a) be a number, where 'x' is a base which is power of 10, and 'a' is deficiency, (may be+ve or -ve). Now

$$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$$

$$= x^2 (x + 3a) + 3xa^2 + a^3$$

$$= x^2 [(x + a) + 2a] + x\ 3\ a^2 + a^3$$

Thus [1] $(x + a)^3 = | (x + a) + 2a | 3a^2 | a^3$ part wise

[2] Add Part wise.

Note - If base is 10^n then second and third part contains n digits.

$$\textit{Ex.1. } 15^3 = (10+5)^3$$

$= |\ 15 + (2 \times 5)\ |\ 3 \times 25\ |\ 125$

$= |\ 25\ |\ 75\ |\ 125$ Base = 10

$= 25\ |\ 5\ |\ 5$

$7\ |\ 2$

1

$= 3375$

$$\textit{Ex.2. } 97^3 = (100-3)^3$$

$= |\ 97 - 2 \times 3\ |\ 3 \times (-3)^2\ |\ (-3)^3$

$= 91\ |\ 27\ |\ 27$ Base = 100

$= 912673$

4.4 The Cube Roots

In this section, we shall study how to find cube root of given perfect cube number whose cube root is two-digit number. This is a special case. We first study the following observations :-

(1) If the last three digit of number are all zeros, then cube root contains only one zero at its unit place.

(2) If Beejank of any number is 1, 8 or 9 then only it is perfect cube number otherwise not.

For example : 8000 is perfect cube because B(8000) = 8.

but 14000 is not a perfect cube because B(14,000) = 5.

(3) If the digit at unit's place of given number is 1, 4, 5, 6 or

9 then cube root of this number has 1, 4, 5, 6 or 9 at its unit's place respectively.

4) If the digit at unit's place of given number is 2, 3, 7 or 8 then cube root of this number has 8, 7, 3 or 2 (complement of 10) respectively at its unit's place.

5) Remember the following table :-

Number	1	2	3	4	5	6	7	8	9	10
Cube	1	8	27	64	125	216	343	512	729	1000
Beejank	1	8	9	1	8	9	1	8	9	1

4.5 *Method of Finding Cube Root*

Steps: -

1) Group the number as three digit group from unit's place and onwards to left.

Examples :-	**Numbers**	**Groups**
a)	6 8 9 2 1	68, 921
b)	1 9 6 8 3	19, 683
c)	1 4 8 8 7 7	148, 877
d)	6 1 4 1 2 5	614, 125
e)	8 0 0 0	8, 000

2) Observe the digit at unit's place of given number and select the digit at unit's place of cube root with the help of observations (1) to (4).

	Number given	Digit at Digit at unit's place of number	Digit at Digit at unit's place of cube root.
a)	6 8 9 2 1	1	1
b)	1 9 6 8 3	3	7
c)	1 4 8 8 7 7	7	3
d)	6 1 4 1 2 5	5	5
e)	8 0 0 0	0	0

3) Locate the nearest lower or equal perfect cube number of left most group number from table (5) and write its cube root. This value is ten's place of required cube root.

	Number given	Left most group	Nearest lower / equal perfect cube	Its cube root.
a)	6 8 9 2 1	6 8	6 4	4
b)	1 9 6 8 3	1 9	8	2
c)	1 4 8 8 7 7	1 4 8	1 2 5	5
d)	6 1 4 1 2 5	6 1 4	5 1 2	8
e)	8 0 0 0	8	8	2

4) The cube root of given number is :

	Number	*Cube Root*
a)	6 8 9 2 1	41
b)	1 9 6 8 3	27
c)	1 4 8 8 7 7	53
d)	6 1 4 1 2 5	85
e)	8 0 0 0	20

5) We check the result by Beejank Method.

Steps: -

1. Find Beejank of given number.
2. Find Beejank of cube of Beejank of answer.
3. If these two Beejankas are equal then the answer is correct.

	Given Number	**Its Beejank**	**Cube Root**	**its Beejank**	**Beejank of cube of Beejank.**
a)	6 8 9 2 1	8	41	5	$B(5^3) = B(125) = 8$
b)	1 9 6 8 3	9	27	9	$B(9^3) = B(729) = 9$
c)	1 4 8 8 7 7	8	53	8	$B(8^3) = B(512) = 8$
d)	6 1 4 1 2 5	1	85	4	$B(4^3) = B(64) = 1$
e)	8 0 0 0	8	20	2	$B(2^3) = B(8) = 8$

Exercise :-4.1 ***Find the Cube Root.***

(1) 64,000 (2) 9261 (3) 830584 (4) 5832

(5) 10648 (6) 103823 (7) 125,000 (8) 3375

(9) 32768 (10) 97336

Answers :-4.1

(1) 40 (2) 21 (3) 94 (4) 18 (5) 22 (6) 47 (7) 50 (8) 15(9) 32 (10) 46

Exercise :-4.2 ***Find the cubes of following numbers.***

[1]. 12 [2] 34 [3] 46 [4] 27

[5] 61 [6] 57 [7] 78 [8] 83

[9] 93 [10] 66

Answers :-4.2

[1]. 1 7 2 8 [2] 3 9 3 0 4 [3] 9 7 3 3 6

[4] 1 9 6 8 3 [5] 2 2 6 9 8 1 [6] 1 8 5 1 9 3

[7] 4 7 4 5 5 2 [8] 5 7 1 7 8 7 [9] 8 0 4 3 5 7

[10] 2 8 7 4 9 6.

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CHAPTER 5.

DIVISION

5.1 (Nikhilam Method)

This method is suitably applied for those divisors which are nearer to the numbers 10^1, 10^2, 10^3, 10^4 ..... which are known as the base numbers.

The method is explained by examples.

Ex. (1) Divide 12311 by 9.

Steps:

[1] Dividend, divisor and difference of divisor 9 from base 10 equal to 1is written as shown in table.

[2] Draw remainder line so that number of digits on its right side should be equal to number of zeros in the base.

9	1 2 3 1	1
1	1 3 6	7
	1 3 6 7	8

[3] Remaining procedure is similar to synthetic division.

(a) Left most digit 1 as it is.

(b) 1 x 1 = 1, 2 + 1 = 3

(c) 1 x 3 = 3, 3 + 3 = 6

(d) 1 x 6 = 6, 6 + 1 = 7

(e) 1 x 7 = 7, 7 + 1 = 8

Thus Quotient Q = 1367 and Remainder R = 8.

Ex. (2). Divide 1 5 1 3 by 9 7

Steps: [1] Divisor is 97, base = 100, Difference = 03.

Note :- Here difference is considered as two-digit number.

[2] The division is performed as shown in the table.

97	1	5	1	3
03		0	3	
			0	15
	1	5	4	18

Ans : Quotient = 15 , Remainder = 4 | 18 = 58 (Adding part wise)

Ex. (3) Divide 2 1 3 3 2 1by 8 9 8.

Steps [1] Base 1000.Difference from base 102.

898	2	1	3	3	2	1
102		2	0	4		
			3	0	6	
				6	0	12
	2	3	6	13	8	13

= 1393 (Adding part wise).

But as the divisor is 989 , the remainder can not be more than 898. Therefore repeat the same process for 1393.

898	1	3	9	3
102		1	0	2
	1	4	9	5

Ans :Remainder = 495, Quotient = 236 + 1 = 137.

Ex. (4) Find 3 0 0 0 0 0 ÷ 8998

8998	3 0	0 0 0 0
1002		
	3	0 0 6
		3 0 0 6
	3 3	3 0 6 6

Thus Remainder = 3066 and Quotient = 33.

5.2 Advantages of Vedic Method

(1) The current method is of a trial and error type as well as tedious and time consuming also. But in Vedic method, the bigger the digits the smaller will be the required compliment (from 9 or 10 as the case may be).

(2) There is no subtraction at all.

(3) We have to multiply by single digit to single digit. In other words

$9 \times 9 = 81$ is a multiplication that one has to perform.

5.3 Limitations of Nikhilam Method

(1) Nikhilam method is not suitable for divisor having smaller digits because

compliment will have bigger digits, which is not favorable to speedy computations .

(2) For smaller divisors "Paravartya" formula is used which is discussed below

Paravartya Method:-

5.4 Paravartya method is used when the divisor is greater than but nearer to the base.

Sutra : Paravartya yojayet

(Transpose and apply)

Ex. (5) 1 3 4 ÷ 1 2

Here divisor is 12 which is nearer to 10 hence assume base as 10.

Now by sutra we get - 2 as modified divisor. The dividend, divisor, quotient and remainder are written as shown in the table.

12	1	3	4
$\bar{2}$		$\bar{2}$	$\bar{2}$
	1	1	2

Thus Quotient = 11 and Remainder = 2.

Ex. (6) Find. 2 3 9 4 7 9 ÷ 1 1 2 0 3

Base = 10000 = 10^4

1 1 2 0 3	2	3	9	4	7	9
$\bar{1}\ \bar{2}\ 0\ \bar{3}$						
		$\bar{2}$	$\bar{4}$	$\bar{0}$	$\bar{6}$	
			$\bar{1}$	$\bar{2}$	$\bar{0}$	$\bar{3}$
	2	1	4	2	1	6

Thus Quotient = 21 and Remainder = 4216.

Urdhva Tiryak Method

5.5 Now we will study the Vedic method of division of numbers, which is called Straight division method. This method can be described as one of the best contribution of Swamiji

SUTRAS :- {1}ध्वजांक:

{2}ऊर्ध्वतिर्यग्भ्याम्

5.6 Meaning of Flag digit / s (FT)
and Operative divisor (O.div.)

1. For single digit divisor, FT does not exist and O.div. is given digit.

2. For double-digit divisor, FT is digit at unit's place and O.div. is digit at ten's place.

3. For triple and higher digit divisor, O.div. is digit at highest place and remaining digits are considered as flag digits.

Method of division

5.7 Divisor with no flag digit

Note :-

ND = Net dividend, GD = Gross dividend, Q = Quotient, R = Remainder.

Ex. (7). Divide 2449 by 6.

The table of operations is as follows.

R. line			2	0	4	1	4
O.div.	6						
Dividend		2	4	4	9	0	0
Q =		0	4	0	8	1	6

Steps [1] 2 / 6 gives Q = 0 and R = 2, then GD = 24 and ND = 24.

[2] 2 4 / 6 gives Q = 4 and R = 0, then GD = 4 and ND = 4.

[3] 4 / 6 gives Q = 0 and R = 4, then GD = 49 and ND = 49

[4] 49 / 6 gives Q = 8 and R = 1.

Thus **Q = 408 and R = 1.**

To find the answer up to two decimal places (say), we insert two zeros in dividend and same procedure is continued.

Thus **Q = 408 . 16**

5.8 Divisor with one FT.

Ex. (8). Divide 45304 by 42.

The table of operations is as follows.

R. line			0	3	5	4
F T	2					
O.div.	4					
Dividend		4	5	3	0	4
Q =		1	0	7	8	

Steps

[1] 4 / 4 gives Q = 1 and R = 0, then GD = 05 and ND = 5 - (2 x 1) = 3

[2] 3 / 4 gives Q = 0 and R = 3, then GD = 33 and ND = 33 - (2 x 0) = 33

[3] 33 / 4 gives Q = 7 and R = 5,

(Q and R are selected so as to avoid negative ND)

then GD = 50 and ND = 50 - (2 x 7) = 36.

[4] 36 / 4 gives Q = 8 and R = 4, [Note Q and R] then GD = 44.

[5] 44 - (2 x 8) = 28.

Thus **Q = 1078 and R = 28.**

Ex. (9). Divide 735427 by 58.

The table of operations is as follows.

We modify the divisor as 58 = $6\overline{2}$

R. line			1	3	3	4	2
F T	$\overline{2}$						
O.div.	6						
Dividend		7	3	5	4	2	7
Q =		1	2	6	7	9	

Steps: -

[1] 7 / 6 gives Q = 1 and R = 1, then GD = 13, ND =

$13 - (\bar{2} \times 1) = 15.$

(We add the cross product of FT and Q digit because FT is $\bar{2}$)

[2] 15 / 6 gives Q = 2 and R = 3, then GD = 35, ND

$= 35 - (\bar{2} \times 2) = 39.$

[3] 39 / 6 gives Q = 6 and R = 3, then GD = 34, ND

$= 34 - (\bar{2} \times 6) = 46.$

[4] 46 / 6 gives Q = 7 and R = 4, then GD = 42, ND

$= 42 - (\bar{2} \times 7) = 56.$

[5] 56 / 6 gives Q = 9 and R = 2, then GD = 27, ND
$= 27 - (\bar{2} \times 9) = 45.$

Thus **Q = 12679 and R = 45.**

5.9 Divisor with two or more FTs

When flag digit place contains two or more digits, we subtract the cross product of flag digits with quotient digits from GD to find ND.

The procedure is explained by following two examples.

Ex. (10). Divide 1516924 by 631.

The table of operations is as follows.

R. line			1	3	1	2	1	0
F T	3 1							
O.div.	6							
Dividend		1	5	1	6	9	2	4
Q =		0	2	4	0	4		

(Here flag digit place contains two digits)

Steps:

[1] 1/6 gives Q=0 and R=1, then GD=15, ND=15-(0 x 3)=15.

[2] 15 / 6 gives Q = 2 and R = 3, then GD = 31,

ND = 31 - [(0 x 3) + (3 x 2)] = 25.

[3] 25 / 6 gives Q = 4 and R = 1, then GD = 16 and

ND = 16 - [(2 x 1) + (3 x 4)] = 02.

[4] 02 / 6 gives Q = 0 and R = 2, then GD = 29.

ND = 29 - [(4 x 1) + (3 x 0)] = 25..

[5] 25 / 6 gives Q = 4 and R = 1, then GD = 12.

ND = 12 - [(0 x 1) + (3 x 4)] = 0.

Step 6: 04 - [(4 x 1) = 0.

Thus **Q = 2404 and R = 0.**

Ex. (11). Divide 3141. 53 by 2641

The table of operations is as follows.

$2641 = 3\bar{4}41$

Remainder line.			0	2	0	1	3
F T	$\bar{4}41$						
O.div.	3						
Dividend		3	1	4	1	5	3
Q =		1	1	8	9	5	

Steps

[1] 3 / 3 gives Q = 1 and R = 0, GD = 01, ND=01-(1 x 4) = 5.

[2] 5 / 3 gives Q = 1 and R = 2, GD = 24,

ND = 24 - [(1 x 4) + (1 x 4)] = 24.

[3] 24 / 3 gives Q = 8 and R = 0, then GD = 01,

ND = 1 - [(8 x 4) + (1 x 4) + (1 x 1)] = 28.

[4] 28 / 3 gives Q = 9 and R = 1, then GD = 15,

ND = 15 - [(9 x 4) + (8 x 4) + (1 x 1)] = 18.etc.

Thus **Q = 1.1895** .....

EXERCISE 5.1

(A) Divide by Nikhilam method

[1] 123 by 9 [2] 135 by 7 [3] 12123 by 88

[4] 13532 by 988

(B) Divide by Paravartya method

[1] 214 by 14 [2] 11111 by 1012 [3]

(C) Divide by Urdhva Tiryak method.

[1] 573 by 7 [2] 5325 by 40 [3] 823.52 by 8

[4] 4853 by 23 [5] 1717632 by 48 [6] 48985 by 83

[7] 5184 by 324 [8] 55753 by 439 [9] 1064321 by 743

[10] 46315 by 1054 [11] 97531 by 1818

[12] 879540 by 3210 [13] 975311 by 16333

[14] 24379159 by 7143

(Find remainders and hence answer to three decimal places

wherever possible)

ANSWERS

(A) [1] Q = 13, R = 6 [2] Q = 19, R = 2 [3] Q = 137, R = 67

[4] Q= 13, R=688 (B) [1] Q =15, R=4 [2] Q=10, R= 991

(C) [1] Q = 81, R = 6 [2] Q = 133, R = 5 [3] Q = 102.94

[4] Q = 211, R = 0 [5] Q=35784, R= 0 [6] Q = 590, R = 15

[7] Q = 16, R=0 [8] Q= 127, R = 0 [9] Q = 1432, R = 345

[10] Q = 43. 9421 [11] Q = 53.,647 [12] Q = 274, R = 0

[13] Q = 59, R = 11664 [14] Q = 3413.0139

5.10 Special Cases

The numbers can be simultaneously added, subtracted and then divided by given number, by applying the procedure similar to straight division method. Only difference is that addition or subtraction is worked out from left to right with proper adjustment of carry digit .

Illustrative examples: -

Ex. 12 :— Find (64 + 43 + 25) / 4.

Ans : We write				<u>Steps</u>
		6	4	1] $6 + 4 = 2 = 12$
	+	4	3	2] $12/4 : Q = 3, R = 0.$
4/	+	2	5	3] $4 + 3 + 5 = 12$
				4] $12/4 : Q = 3, R = 0.$
		3	3	

Ans :- **Q = 33 , R = 0.**

Ex. 13 :- Find (532 + 371 + 625 - 294) / 7 .

Ans : We write

		50	40			
		5	3	2		
	+	3	7	1		
	+	6	2	5		
7/	-	2	9	4	2	
		1	7	6		

Steps

1] 5 + 3 + 6 - 2 = 12.

2] 12 / 7 : Q = 1 R = 5.

3] carry '5' to next right column gives ' 50 '.

4] 50 + 3 + 7 + 2 - 9 = 53

5] 53 / 7 : Q = 7 R = 4.

6] 40 + 2 + 1 + 5 - 4 = 44.

7] 44 / 7 : Q = 6 R = 2.

Ans : Q = 176 , R = 2 .

Remark **:** - Above method is very effective in finding mean or average of given set of the numbers

Ex. 14 :- Find the value of (5319 - 2523 + 1608 - 1324) / 51.

Ans . -

			30	10	10		
		5	3	1	9	50	20
	-	2	5	2	3		
1	+	1	6	0	8		
5	-	1	3	2	4		
		0	6	0	3	9	2

Steps

[1] 5 - 2 + 1 - 1 = 3 , 3 / 5 gives Q = 0 , R = 3

[2] 30 + 3 - 5 + 6 - 3 = 31

[3] 31 - (1 X 0) = 31 31 /5 gives Q = 6 , R = 1

[4] 10 + 1 - 2 + 0 - 2 = 7,

[5] 7 - (1 X 6) = 1 1 / 5 gives Q = 0 , R = 1

[6] 10 + 9 - 3 + 8 - 4 = 20

Thus Q = 60 , and R = 20.

If division is continued we get

[7] 20 - (1 X 0) = 20 20 / 5 gives Q = 3 , R = 5 (Note)

[8] 50 - (1 X 3) = 47 47 / 5 gives Q = 9 , R = 2

[9] 20 - (1 X 9) = 11 11 / 5 gives Q = 2 , R = 1, etc.

The answer is **60 . 392**

EXERCISE 5.2

Find the value of :—

[1] (57 + 81)2 [2] (6945 + 5937 + 3453)5.

[3] (428 + 213 - 378)4 [4] (4538 + 3250 - 2389)5

[5] (35 + 63 + 40)23 [6] (457 +258 - 179)22

[7] (427 + 1309 -532 -221 + 947)83

[8] (1012 + 831 - 317 + 2454)29 .

[9] Find the mean of :— 54, 46, 48, 50, 47 up to two decimal places.

[10] Find the mean of : - 134, 132, 137, 136, 138, 133

{ Find answers up to three decimal places, wherever possible }

ANSWERS

[1] 69	[2] 3267
[3] Q = 65 , R = 3 , 65 . 75	[4] 1079.8
[5] Q = 6 , R = 0	[6] Q = 24. 3636
[7] Q = 23 , R = 21	[8] Q = 137 , R = 7
[9] 49	[10] 135.

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CHAPTER 6.

DIVISIBILITY

6.1 We have the standard rules for divisibility by 2, 3, 4, 5, and 11. But our secondary school textbooks do not suggest any divisibility tests to decide whether an integer is divisible by 7, 13, 17, 19, 23, 29, 31 etc. Vedic Mathematics gives method to determine beforehand whether a certain given number (however large it may be) is divisible by given divisor.

To understand the method one should get acquainted with the following concepts,

1) *वेष्टनांक, वेष्टनांक क्रिया, वेष्टनांक क्रिया फल.*

that is, osculator, osculation and osculation result.

2) Different types of osculators :-

P :- Positive Osculators.

Q :- Negative Osculators

P_1 :- + ve Osculator covering one digit.

P_n :- + ve Osculator covering n digits.

Q_1 :- - ve Osculator covering one digit.

Q_n :- - ve Osculator covering n digits.

u : digit in unit's place.

T : digit in ten's place.

6.2 About Osculators (Veshtanank).

Osculator is parameter obtained from divisor, which is

the basic requirement for divisibility. Osculators do not exist for even integers.

To determine positive Osculator of a given divisor.

[1] Multiply the divisor by such a minimum number, which yields nine at the unit's place in the product.

[2] Add one to the product.

[3] Omit zero appearing at unit's place.

[4] The remaining digits gives Osculator of that divisor.

Ex.1. To find Osculator of 7 (the divisor)

(7 x 7) + 1 = 50 Osculator of 7 is 5.

Ex.2. Osculator of 23 is 7. because (23 x 3) + 1 = 70.

Ex.3.. As (47 x 7) + 1 = 330 Osculator is 33.

6.3 *To determine negative Osculator of a number (divisor).*

[1] Multiply the divisor by such a minimum number, which yields 1 at the unit's place in the product.

[2] Omit the " digit 1 " appearing at the unit's place.

[3] The remaining digits give Osculator of divisor.

Ex.1. To find Osculator of 17.

We have 17 x 3 = 51 Osculator of 17 is 5.

Ex.2.. The Osculator of 13 is 9 as 13 x 7 = 91 (Omit one).

6.4 *List of positive Osculators (P)*

1) P for 9, 19, 29, 39 etc. (ending in 9) is 1, 2, 3, and 4 respectively.

2) P for 3, 13, 23, 33 etc. (ending in 3) is 1, 4, 7, and 10 respectively.

3) P for 7, 17, 27, 37 etc. (ending in 7) is 5, 12, 19, and 26 respectively.

List of negative Osculators (Q)

1) Q for 11, 21, 31, 51 etc. (all ending in 1) is 1, 2, 3, and 5 respectively

2) Q for 7, 17, 27, 37, 47 etc. (all ending in 7) is 2, 5, 8, 11, 14 resp.

3) Q for 3, 13, 23, 43 etc. (all ending in 3) is 2, 9, 16, 23 resp.

4) Q for 9, 19, 29, 49 etc. (all ending in 9) are 8, 17, 26, 35 etc.

6.5 *Important Features of Osculators*

1) P + Q = D , D : Divisor.

2) The multiples of 2 and 5 are inadmissible for the purpose of Osculator

3) For divisors ending in 3 $P < Q$.

4) For divisors ending in 7 $Q < P$.

6.6 *Test for divisibility :-*

Osculation process and Osculation result for P

Ex.1) Is 21 divisible by 7, Notation : Is 7/21? (useVeshtanank method)

Step 1) Multiply the last digit (1) by Osculator (of 7) and add the product to the previous digit. The process known as "Osculation" and the number so obtained known as "Osculation result ".

Osculator of 7 is 5 (i.e. P = 5)

Osculation result = P.u + T

= (5 x 1) + 2

= 7

Here Osculation Result = Divisor.

21 is divisible by 7.

Solved examples : Ex. Is 19 / 95 ?

Here D = 19 , P = 2, u = 5, T = 9 (Previous digit).

Osculation Result = P. u + T = (2 x 5) + 9

= 19

= 19 Divides 95

Rule :-

1) If Osculation Result = Divisor, then the number is

divisible.

2) If we get Osculation Result a bigger number then Osculate the result again till we get a smaller number.

3) If the Osculation Result is either zero or equal to divisor or divisible by divisor then the original number is perfectly divisible.

4) R_1, R_2, R_3, ——— denote Osculation results at intermediate steps.

Ex. 3) Test : 29 3 2 8 9 6 P = 3

3 2 8 9 6

27 8 31 27

Steps 1) Osculate 6, we have

$$R_1 = Pu + T$$

$$= 3\ 6 + 9 = 27$$

2) $R_2 = 3\ 7 + 2 + 8 = 31$

3) $R_3 = 3\ 1 + 3 + 2 = 8$

4) $R_4 = 3\ 8 + 3 = 27$

Last Osculation Result $R_4 = 27$.

R_4 not divisible by D

3 2 8 9 6 Not Divisible by 29.

We have another way to write the Osculation Results.

We denote them by S_1, S_2, S_3 (complete sum).

Alternative Way :- 2 9 3 2 8 9 6

	3 2 8 9 6	1) P = 3
+	1 8	2) u x P = 6 x 3 = 18
	3 3 0 7	add to previous continue
+	2 1	3) Osculation Results are :
	3 5 1	$S_1 = 3307$
+	3	$S_2 = 351$
	3 8	$S_3 = 38$
+	24	$S_4 = 27$
	27	

Last Osculation Result is 27, Not divisible by divisor (29)

∴ Not divisible.

Ex. 4) 59 1 9 1 5 7 3

1 9 1 5 7 3

59 49 46 37 25

D = 59

P = 6

$R_1 = P\ u + T$

$= 6\ 3 + 7 = 25$

$R_2 = 6\ 5 + 2 + 5 = 37$

$R_3 = 6\ 7 + 3 + 1 = 46$

$R_4 = 6\ 6 + 4 + 9 = 49$

$R_5 = 6\ 9 + 4 + 1 = 59$

Last Osculation $R_5 = D$

Divisible by 59.

Alternative Way :- 59 1 9 1 5 7 3

$$\begin{array}{rl}
 & 1\ 9\ 1\ 5\ 7\ 3 \\
+ & 1\ 8 \\
\hline
 & 1\ 9\ 1\ 7\ 5 \\
+ & 3\ 0 \\
\hline
 & 1\ 9\ 4\ 7 \\
+ & 4\ 2 \\
\hline
 & 2\ 3\ 6 \\
+ & 3\ 6 \\
\hline
 & 5\ 9
\end{array}$$

1) $D = 59$

2) $P = 6$

3) $S_1 = 19175$

4) $S_2 = 1947$

5) $S_3 = 236$

6) $S_4 = 59$

Osculation Results

Here $S_4 = D$

Divisible by 59.

Ex. 5) Test if 179 7 1 4 5 5 0 1

7	1	4	5	5	0	1
179	109	6	20	150	18	

$D = 179$

$P = 18$

$R_1 = 1 \ 18 + 0 = 18$

$R_2 = 8 \ 18 + 1 + 5 = 150$

$R_3 = 0 \ 18 + 15 + 5 = 20$

$R_4 = 0 \ 18 + 2 + 4 = 6$

$R_5 = 6 \ 18 + 1 = 109$

$R_6 = 9 \ 18 + 10 + 7 = 179$

$R_6 = D$ Divisible by 179.

6.7 Osculation Process and Osculation Results for Q.

Ex. 1) Is 21 divisible by 7 ?

0 Here $D = 21$

$Q = 2$

$R_1 = Q \ u - T$

$= 2 \times 1 - 2$

$= 0.$

Osculation Result = 0
Divisible.

Ex. 2) 6 1 1 9 5 8 1

Here Osculator is - ve.

$$R = Q.u \quad T$$

<u>Here the sign is alternate plus and minus</u>

$$R_1 = Q.u - T$$

$$R_2 = Q.u + T$$

$$R_3 = Q.u - T$$

Ans :- 1 9 5 8 1

0 -51 -7 $\bar{2}$

1) Digits in the dividend marked alternate + ve , - ve with bars overhead.

2) $Q = 6.$

3) $R_1 = Q.u - T$
$= 6 \times 1 - 8 = -2$

4) $R_2 = 6(-2) + 5 = -7.$

5) $R_3 = 6(-7) - 9 = -51$

6) $R_4 = -(6 \times 1 - 5) + 1 = 0.$

Last Osculation Result = 0.

Divisible by 61.

Ex. 3) 67 1 0 1 7 1 2 0 3

1 0 1 7 1 2 0 3

0 -10 -101 -5 -81 -4 60

1) $Q = 20$

2) $R_1 = 20 \times 3 - 0 = 60$

3) $R_2 = 20 \times 0 - 6 + 2 = -4$

4) $R_3 = 20 \times (-4) - 1 = -81$

5) $R_4 = -(20 \times 1 - 8) + 7 = -5$

6) $R_5 = 20 \times (-5) - 1 = -101$

7) $R_6 = -(20 \times 1 - 10) + 0 = -10$

8) $R_7 = -(20 \times 0 - 1) - 1 = 0.$

Last Osculation Result = 0.

Divisible by 67.

Ex. 4) Alternative Way to write

1 4 1 4 8 9 8 8 5 7 Q = 1 4

4 8 9 8 8 5 7

- 9 8

4 8 9 7 8 7

- 9 8

4 8 8 8 0 At the stage when we get 488,

- 0 we confirm " Not divisible ".

4 8 8

- 1 1 2

- 6 4

Ex. 5) $141|4898857$ $Q = 14$

$4 \quad \overline{8} \quad 9 \quad \overline{8} \quad 8 \quad \overline{5} \quad 7$

$94 \quad 87 \quad 37 \quad 2 \quad 41 \quad 93$

Not Divisible.

6.8 Complex Multiplex Osculators (i.e. P_n , Q_n)

The cases so for dealt with were of simple type, involving small divisors and consequently small osculators. What about those wherein bigger numbers being the divisors, the osculators are bound to be correspondingly large ?. This difficulty can be overcame by formulating a scheme of groups of digits, which can be osculated not as individual digits but in a lump.

Examples of Multiplex Osculation :-

1) 371 Osculated by 4 for 2 digits at a time [i.e. The Osculator is of the type P_2 or Q_2].

Osculation Results :-

1) For $P_2 = 71 \times 4 + 3$.

2) For $Q_2 = 71 \times 4 - 3$.

6.9 Categories of Divisors and their Osculators

1) Those divisors, which end in Nine or series of Nine, come under positive Osculators.

E.g.

Osculator for 2 9, $P = 3$ (Covering one digit)

Osculator for 2 9 9, $P_2 = 3$ (Covering 2 digits).

Osculator for 1 2 9 9 9, $P_3 = 13$ (Covering 3 digits).

2) Those divisors, which terminate in or contain series of zeros ending in 1.These come under negative Osculator.

E.g.

Osculator for 5 1, $Q = 5$ (Covering one digit).

Osculator for 5 0 1, $Q_2 = 5$ (Covering 2 digits).

Osculator for 5 0 0 1, $Q_3 = 5$ (Covering 3 digits).

Osculator for 1 2 0 0 0 0 0 0 1, $Q_7 = 12$ (Covering 7 digits).

3) Those divisors which by suitable multiplication yield a multiple either of the two types :

1) Ending in 9 or a series of nines.

2) Ending in unity or a series of zeros ending in unity.

E.g.

1) Divisor = 8 5 7 — As 8 5 7 x 7 = 5 9 9 9

$P_3 = 6$.

2) Divisor = 4 3 — As 4 3 x 7 = 3 0 1

we have $Q_2 = 3$.

or As $43 \times 3 = 129$

we have $P_1 = 13$.

3) Divisor = 229   As $229 \times 131 = 29999$

$P_4 = 3$.

4) Divisor = 421   As $421 \times 19 = 7999$

$P_3 = 8$.

To find suitable multiplier - to determine P_n / Q_n

E.g. 1) Find P_n for 142857.

To find Pn we multiply by suitable multiplier which yield a product ending in 9 or series of nine.

Multiply by 7.

$$\begin{array}{r} 142857 \\ \times \quad 7 \\ \hline 999999 \\ + \quad 1 \\ \hline 10{,}00000 \end{array}$$

$P_5 = 10$

E.g. 2). To find P_n for 76923

$$\begin{array}{r} 76923 \\ \times \quad 3 \\ \hline 230769 \end{array}$$

1) On multiplying by 3 we get 6 at tens place.

2) To get 9 at this place three should be added to 6, for that multiply by 13 instead of 3.

```
  7 6 9 2 3
x       1 3
-----------
2 3 0 7 6 9
7 6 9 2 3 x
-----------
9 9 9 9 9 9
```

3) 7 6 9 2 3 x 1 3 = 9 9 9 9 9 9.

$P_5 = 10$

Ex. : Is 1 0 6 6 5 6 8 7 4 2 6 9 divisible by 4 9 9 ?

1) D = 4 9 9, $P_2 = 5$

2) Osculator is P_2 this means that we have to split the given dividend in to 2 digit groups and Osculate by 5. Thus

10	66	56	87	42	69
499	497	186	525	387	

$R_1 = 69\ 5 + 42 = 3\,8\,7$

$R_2 = 87\ 5 + 3 + 87 = 5\,2\,5$

$R_3 = 25\ 5 + 5 + 56 = 1\,8\,6$

$R_4 = 86\ 5 + 1 + 66 = 4\,9\,7$

$R^{-}_5 = 97\ 5 + 4 + 10 = 4\,9\,9$

The last osculation result = D.

Divisible by 499.

Alternative Way :-

10 66 56 87 42 69	Osculate a group of 2
3 45	as $P_2 = 5$.
10 66 56 90 87	1) 69 x 5 = 3 4 5
+ 4 3 5	2) 87 x 5 = 4 3 5
10 66 61 25	3) 25 x 5 = 1 2 5
+ 1 25	4) 86 x 5 = 4 3 0
10 67 86	5) 97 x 5 = 4 8 5
+ 4 30	
14 97	
+ 485	
499	

Last Osculation Result = D

Divisible by 499.

***Ex.* 2)** Is 1 2 6 1 4 3 6 2 2 9 3 2 divisible by 4 0 1?

$\overline{12}$ 61 $\overline{43}$ 62 $\overline{29}$ 32

\- 16 400 185 458 99

1) D = 4 0 1

Osculator $Q_2 = 4$

2) Osculate a group of two digits.

3) Split D in to group of 2.

4) Mark the groups alternately +ve and -ve by giving bar over the group from right to left.

5) Osculate 32 by 4.

$R_1 = 32 \times 4 - 29 = 99$

$R_2 = 99 \times 4 + 62 = 458$

$R_3 = 58 \times 4 - 4 - 43 = 185$

$R_4 = 85 \times 4 - 1 + 61 = 400$

$R_5 = 00 \times 4 - 4 - 12 = -16$

Last Osculate Result = - 1 6

Not Divisible.

Alternative Way :-

```
        12   61   43   62   29   32
                            - 1 2 8
   --------------------------------
        12   61   43   61   01
                       - 0 4
   --------------------------------
        12   61   43   57
                 - 2 2 8
   --------------------------------
        12   59   15
            - 6 0
   --------------------------------
        11   99
   - 3 9 6
   --------------------------------
   - 3 8 5
```

Not Divisible.

6.10 *OSCULATION AND DIVISIBILITY*

To understand the algebraic background of Osculation process and divisibility ; let us review the Osculation process.

E.g. To test whether 6 1 1 is divisible by 1 3. We follow the steps given below.

1) Determine Osculator of 13.
$13 \times 3 + 1 = 40$ Osculator of 13 is 4 i.e. $P = 4$.

2) Multiply digit in unit's place of dividend by P, $4 \times 1 = 4$.

3) Add this product to Ten's place (left hand) of dividend.
$4 + 61 = 65$

4) This sum known as Osculation Result = P u + T
= $4 \times 1 = 61$
= 65.

5) See whether Osculation Result is equal to or divisible by divisor.

65 is divisible by 13, $\therefore$ 611 is divisible by 13.

6) If we get a bigger Osculation result then we repeat the Osculation process for the Osculation Result (65).

7) The whole process is written in short as follows :.

	6 1 1	OR	6 1 1
+	4		2 6 5
	6 5		

(76)
+ 2 0 The last Osculation Result 26 is divisible by 13.

2 6 $\therefore$ 6 1 1 is divisible by 1 3.

6.11 Analysis

1) We know that :-

a, b, c are three integers. If a divides b and a divides b + c then a divides c.

2) Using this property of divisibility .....

To see whether 6 1 1 is divisible by 1 3,

add 13 x 3 = 3 9 to 6 1 1 ......

..... we get 6 1 1 + 3 9 = 6 5 0

(b + c)

3) Now we see whether 650 is divisible by 13. For that consider only 65.

Here 1 3 divides 6 5

$\therefore$ 1 3 divides 6 5 0 i.e. (611 + 39).

$\therefore$ 1 3 divides 6 1 1

4) Now the question arises, why we add three times 13 to 611.

i.e. (611 + 3 x 13)

5) The observation tells us that this particular addition gave us zero at unit's place. (zero at unit's place of 650) Now to determine divisibility we think of only 65 and not 650. (This simplifies our task).

6) In short the Sum is (Dividend + Divisor x Some Integer)

611 + 13 x 3

Here select such a minimum integer, which gives zero at unit's place of the sum.

7) Then the number obtained by omitting zero is considered for divisibility.

8) If the (65) above sum is bigger then divisor than we continue the process.

6.12 Relation with Osculator

1) Suppose Dividend = $10x + y$

Divisor = 29

Osculator P = 3.

2) To see whether 29 divides $10x + y$, add $29y$ we get,

$$(10x + y) + 29y$$

$$= 10x + 30y$$

$$= 10(x + 3y)$$

$$= 10(x + Py)$$

= 10 [Digit in ten's place + Osculator digit in unit's place]

= Osculation Result.

From this we observe that if osculation result i.e. $(10x + y) + 29y$ is divisible by 29 then $10x + y$ is also divisible by 29.

Note :-

1) $10x + y$ need not be considered as two digit number. The given number is written in this form simplify to show the unit place digit separately.

2) The above explanation is for positive Osculator. Such explanation can be given for negative osculators. For this, the following rule is useful.

a, b, c are three integers such that, a divides b and a divides (b - c) then a divides c.

Alternative (For positive Osculator)

Dividend $10x + y$

Divisor a

Select an integer b , which yields 9 at the unit's place of the product $a \times b$.

Osculator of a $= (a \times b + 1) / 10$

i.e. $P = (a \times b + 1) / 10$

Now to see whether a divides $10x + y$ add $a \times b \times y$ to it.

(Note, $a \times b \times y$ is divisible by a.)

We get $(10x + y) + a \times b \times y$

$= 10x + y(ab + 1)$

$$= 10\,x + 10\,y\,(ab + 1)\ /\ 10$$

$$= 10\,x + 10\,py$$

$$= 10\,(x + Py)$$

= 10 [Digit in ten's place + Osculator x unit's place digit]

= ∴ Osculation Result.

Here if Osculation Result i.e. $(10\,x + y) + aby$ is divisible by a, then a divides $10\,x + y$. This explains the Osculation process.

6.13 For negative Osculator Q

The same explanation can be provided for negative Osculator with little change viz.

1) Select an integer b which yield 1 at the unit's place of the product a x b.

   Osculator $Q = a \times b - 1\ /\ 10$

2) Instead of adding aby here we subtract.

   Rest is same.

Exercise :-

A) Test the following divisibility.

   1) 5377 2) 1218 3) 153313.

4) 2131589 5) 31476108.

B) Test whether the number

1) 3 9 9 2 1 is divisible by 7.

2) 5 9 1 5 is divisible by 1 3.

❖ ❖ ❖

CHAPTER 7.
BINARY NUMBERS

7.1 The name decimal system is derived from the fact that it contains ten digits namely 0,1,2, ————, 9. Hence the base of the system is ten. However when the system contains two digits 0 and 1 it is called binary system only and the base here is 2.

In this chapter we aim to study the basic operations in binary system with the help of Vedic sutras.

7.2 Conversion 0f Decimal Numbers To Binary Numbers.

Case [1] Whole numbers.

Ex. 1 Convert 19 into binary number.

Steps:-

[1] Divide 19 by 2. Write the quotient 9 as left most digit in the quotient line, and write the remainder 1 as right most digit in remainder line as shown.

This remainder is called Least Significant Digit (LSD).

2 19

9 | 4 | 2 | 1 | 0 Quotient line.

---

1 | 0 | 0 | 1 | 1 Remainder line

---

[2] Now divide 9 by 2 and write the quotient 4 to right of first quotient 9 in quotient line and remainder1 to left of first remainder 1 in remainder line as shown. The procedure is repeated till we get zero as quotient, and remainder 1 is written in remainder line. This last remainder is called Most Significant Digit (MSD).

Thus decimal number 19 is equivalent to 10011 in binary system.

Case [2] :- Fractional decimal number.

Steps:-

[1] Multiply given fraction by 2. Select first carry as M.S.D.

[2] The remaining fraction is again multiplied by 2 with carry as second binary digit. The procedure is continued till we get zero as fractional digit or extent of accuracy after binary point.

Ex. 2 Convert 0.32 into binary number.

		Carry
Ans.	0.32 X 2 = 0.64	0
	0.64 X 2 = 1.28	1
	0.28 X 2 = 0,56	0
	0.56 X 2 = 1.12	1
		and so on.

Hence the binary number is **0.0101**

Ex. 3. Convert 28.73 in to binary number.

Ans. For whole number 28 we the chart as follows

2 28

14 | 7 | 3 | 1 | 0

---

1 | 1 | 1 | 0 | 0.

For 0.73 we write

0.73 X 2 = 1.46	1
0.46 X 2 = 0.92	0
0.92 X 2 = 1.84	1
o.84 X 2 = 1.68	1 etc.

Thus the Binary number is **11100.10111**

7.3 Conversion of Binary Numbers to Decimal Numbers.

The binary number is weighted number. For the digits on the left of binary point the weights are 2^0, 2^1, 2^2, 2^3, —— etc from

right to left and for the digits on the right part of the binary number the weights are 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}—— etc from left to right. We add the product of the bit and its weight.

Ex. 4 Convert binary number (1 1 0 . 1 0 1) into decimal number.

Ans :- We write the structure as follows :-

$$\begin{aligned} &1\,1\,0\,.\,1\,0\,1 \\ &0 \times 2^{0} = 0.0000 \\ &1 \times 2^{1} = 2.0000 \\ &1 \times 2^{2} = 4.0000 \\ &1 \times 2^{-1} = 0.5000 \\ &0 \times 2^{-2} = 0,0000 \\ &1 \times 2^{-3} = 0.1250 \end{aligned}$$

Adding RHS we get 6.6250

The decimal number is **6.6250.**

7.4 Addition of binary numbers

Sutra: - एकाधिकेन पूर्वेण ekadhikena purvena

(by one more than the previous)

The basic rules for addition is :-

0+0=0, 0+1=1, 1+ 0 = 1, 1 + 1 = 1 0 (where 1 is carry)

And rules for carry bits are : -

1+0 +0 =0 1 with carry 0, 1+0 + 1 = 1 0 with carry 1

1+1 + 0 =1 0 with carry 1, 1+1 + 1 = 1 1 with carry 1

Notation :- *Ekadhika* of the binary digits are denoted as 1*, 0*.

and meaning is : 0* = 0 + 1 =1,1* = 10.

Method :- Write Ekadhika (*) as carry 1, whenever it occurs while adding the binary numbers, in left column

Ex.5 Find 1 0 1 1 0 + 1 1 1 0 1 Steps:-

Ans. We write 1* 0* 1 1 0 0 + 1 = 1, 1+0 = 1,

\+ 1 1 1 0 11 + 1 = 1 0 with carry 1

——————— $0^* = 1$, $1+1 = 10$, carry 1

11 0 0 1 1 $1^*+1 = 10 + 1 = 11$

Ex.6 Find 1 1 1 0 1 1 + 1 0 0 1 0 1 + 1 0 0 1 0 + 1 0 0 1

Ans We write as :-

	1	1	1	0	1	1
+	1*	0	0	1	0	1
		1*	0	0	1	0
+			1*	0*	0*	1
	11	1	1	0	1	1

7.5 Subtraction of binary numbers

Sutra: यावदूनम् Yavadunam

(whichever is difference)

Case [1]:- without carry bits

The basic rules are 0 - 0 = 0, 1 - 0 = 1, 1 -1 = 0.

Ex.7. Find 110010 - 10010.

We write as :

```
    1 1 0 0 1 0
  -   1 0 0 1 0
  -------------
    1 0 0 0 0 0
```

Case [2] :- With carry bits

We note the difference of the digits 1,0*, and 1* from the base 2.

Digits	Difference
1	2 - 1 = 1
0*	2 - 0* = 1
1*	2 - 1* = 0

Method :- when 0 - 1 or 0 - 0* or 0 - 1* is to be found out, add difference of the digit to be subtracted in zero, and insert *Ekadhika* (*) in previous left column.

Thus 0 - 1 = 1 and * on any digit of the left column

0 - 0* = 1 and * on any digit of the left column

0 - 1* = 0 and * on any digit of the left column

Ex. 8 Find 1 1 0 0 0 - 1 0 0 1 1.

Answer:					
	1	1	0	0	0
-	1	0*	0*	1*	1
	0	0	1	0	1

Answer:—— **1 0 1**

7.6 Multiplication of binary numbers

Sutra: ऊर्ध्व तिर्यग् भ्याम् Urdhva tiryakbhyam

(Vertically and crosswise)

We explain the method by examples.

Ex.9 Multiply 1 1 1 0 .

We write

1 1

1 0

(1 1) | { (1 0) + (1 1) } | (0 1)

1 | 1 | 0

Ans : — **1 1 0.**

Ex.10 Find (1 0 1) (1 1 0)

We write

1 0 1

1 1 0

1 | 1 | (1 0 + 0 1 +1 1) | 1 | 0

1 | 1 | 1 | 1 | 0

Ans : — **1 1 1 1 0**

Ex.11. (1 1 0 1) (1 0 0 1) + (1 0 1) (1 1 0) - (1 1 1) (1 0 0).

We write

1 1 0 1		0 1 0 1		0 1 1 1
1 0 0 1	+	0 1 1 0	-	0 1 0 0

(1) | (1) | 0+1-1) | ($_1$1+1-1) | (1 +1-1) | (0 +1-0)| 1

1 1 0 $_1$1 1 1 1

Ans. : — 1 1 1 1 1 1 1.

This example shows that the multiplication and addition or subtraction can be performed simultaneously.

EXERCISE 7.1

Set A :- Convert following numbers into binary numbers.

[1] 9 [2] 47 [3] 128 [4] 345 [5] 87 [6] 0. 6832[7] 35.5

Set B : - Convert following binary numbers into decimal numbers.

[1] 1 1 0 1 [2] 1 0 1 0 1 0 [3] 1 0 1 1 0 0 0

[4] 0 . 1 1 1 [5] 1 1 1 1 . 0 1 1 1

Set C:-Find the value of:-

[1] 1 1 0 1 + 1 1 [2] 1 0 0 1 1 1 0 + 1 0 0 0 1 1 0

[3] 1 1 1 . 0 1 1 + 1 1 . 1 0 [4] 1 1 1 0 1 + 1 0 1 1 1

[5] 1 1 0 1 - 1 0 1 [6] 1 1 0 0 0 -1 0 0 1 1 1

[7] 1 1 1 1 1 1 1 - 1 1 1 1 1 - 1 1 1

[8] 1 0 1 1 1 0 0 - 1 1 1 0 1

Set D : - Find the value of

[1] 1 0 1 0 [2] 1 1 1 1 [3] 1 0 0 1 1 1

[4] 1 1 0 1 1 0 0 0 5] 1 0 0 1 1 1 1 1 0 0

[6] (1 0 0 1 0 1) + (1 0 1 1 1 0).

[7] (1 0 0 1 0) + (1 0 1 1 1 1).

[8] (1 0 1 0) - (1 1 0 1 1 1).

ANSWERS 7.1

Set A : — [1] 1 0 0 1 [2] 1 0 1 1 1 1

[3] 1 0 0 0 0 0 0 0

[4] 1 0 1 0 1 1 0 0 1 [5] 1 0 1 0 1 1 1 [6] 0 . 1 0 1 0 1
[7] 1 0 0 1 1. 1

Set B : — 1] 13 2] 42 3] 88 4] 0.875
5] 15. 3775

Set C : [1] 1 0 1 0 [2] 1 0 0 1 0 1 0 0 [3] 1 0 1 0 . 1 1 1
[4] 1 1 0 1 0 0 .
[5] 1 0 0 0 [6] 1 0 1 [7] 1 0 1 1 0 0 1
[8] 1 1 1 1 1 1
Set D : —[1] 1 0 0 [2] 1 0 0 1 [3] 1 1 0 0
[4] 1 1 0 1 0 0 0 [5] 1 0 0 0 0 1 0 1 0 0
[6] 1 1 0 0 1 0 [7] 1 0 1 1 1 1 [8] 1 1 0 0 .

CHAPTER 8

Multiplication And Division of Polynomials

8.1 The multiplication of polynomials plays an important role in mathematics. In this chapter first we shall study this multiplication by Urdhva Tiryak sutra followed by checking methods which are unique features of Vedic methods.

FORMULAE :-

{1} उर्ध्व तिर्यग् भ्याम् - Urdhva tiryakbhyam

(Vertically and Crosswise)

{2} गुणितसमुच्चय: समुच्चयगुणित: (gunitsamuccyah samuccyagunitah)

(Product of the whole is equal to whole of the product)

Method :- The VM method for multiplication of the polynomials is exactly similar to multiplication of numbers by above sutra { 1 } and hence illustrated by following examples.

Ex. 1. Find (3 x + 2) (4 x + 7).

Ans. :- We write as -

$$\begin{array}{rr} & 3x+2 \\ \times & 4x+7 \\ \hline & 12x^2+29x+14 \\ \hline \end{array}$$

Steps

[1] $\{(3)(4)\}x^2 = 12x^2$.

[2] $\{(3)(7)+(2)(4)\}x = 29x$

[3] $\{(2)(7)\} = 14$

Ex. 2. Find $(5x+1)\ (x+2)$.

Ans. :- We write as -

$$\begin{array}{r} 5x+1 \\ x+2 \\ \hline 5x^2+11x+2 \end{array}$$

Steps

[1] $\{(5)(1)\}x^2 = 5x^2$.

[2] $\{(5)(2)+(1)(1)\}x = 11x$

[3] $\{(1)(2)\} = 2$

Ex. 3. Find $(5x+3y+2)(10x-3y-4)$.

Ans :- We write as :-

$$\begin{array}{r} 5x+3y+2 \\ 10x-3y-4 \\ \hline \end{array}$$

$$50x^2+15xy-9y^2-18y-8$$

Steps

[1] $\{ (5)(10) \} x^2 = 50 x^2$

[2] $\{ 5(-3) + 10(3) \} x y = 15 x y$

[3] $\{ 5(-4) + 10(2) \} x + 3(-3) \} y^2 = 0 x - 9 y^2$

[4] $\{ 3(-4) + 2(-3) \} y = 18 y$

[5] $\{ 2(-4) = -8$.

Ex. 4. Find $(4x + 2y + 1)(7x - 3y - 2)$.

Ans :- We write as :-

$$\begin{array}{rl} & 4x + 2y + 1 \\ \times & 7x - 3y - 2 \\ \hline & 28x^2 + 2xy - x - 6y^2 - 7y - 2 \\ \hline \end{array}$$

S'eps

[1] $\{ (4)(7) \} x^2 = 28 x^2$

[2] $\{ 4(-3) + 2(7) \} x y = 2 x y$

[3] $\{ 4(-2) + 1(7) \} x + 2(-3) \} y^2 = -x - 6 y^2$

[4] $\{ 2(-2) + 1(-3) \} y = -7 y$

[5] $\{ 1(-2) = -2$.

Ex. 5. Find $(2x - 3y + 3z - 1)(x + y - 2z + 3)$.

Ans. We write as : — $2x - 3y + 3z - 1$

$\times \quad x + y - 2z + 3$

$2x^2 - xy - xz - 3y^2 + 5x + 9yz - 19y - 6z^2 + 11z - 3$

Steps: -

[1] $\{ 2 \times 1 \} = 2x^2$

[2] $\{ 2 + (- 3) \} xy = - xy$

[3] $\{- 4 + 3\} xz + \{(- 3) y^2 = - xz - 3y^2$

[4] $\{6 - 1 \} x + \{ 6 + 3 \} yz = 5x + 9yz$

[5] $\{ - 9 - 1 \} y + \{ - 6 \} z^2 = - 10y - 6z^2$

[6] $\{ 9 + 2z \} = 11z$

[7] $\{ - 1 \times 3 \} = - 3$

Ex. 6. Find $(x + 2y + 3z + 1)(x + y + 2z + 2)$.

Ans. We write as : — $x + 2y + 3z + 1$

$\times \quad x + y + 2z + 2$

$x^2 + 3xy + 5xz + 2y^2 + 3x + 7yz + 5y + 6z^2 + 8z + 2$

Steps:

[1] $\{1 \times 1\} = 1x^2$

[2] $\{2 + 1\} xy = 3xy$

[3] $\{2 + 3\} xz + 2y^2 = 5xz + 2y^2$

[4] $\{2+1\} x + \{4 + 3\} yz\} = 3x + 7yz$

[5] $\{4 + 1\} y + \{6\} z^2 = 5y + 6z^2$

[6] $\{6 + 2\} z = 8z$

[7] $\{1 \times 2\} = 2$.

Ex. 7. Find $(x^4 + 7x^2 + 3)(x^3 + x - 4)$.

Ans . We write as follows.

$$
\begin{array}{rl}
 & x^4 + 0x^3 + 0x^2 + 7x + 3 \\
\times & 0x^4 + x^3 + 0x^2 + x - 4 \\
\hline
 & x^7 + x^5 + 3x^4 + 3x^3 + 7x^2 - 25x - 12 \\
\hline
\end{array}
$$

Steps:

[1] $\{1 \quad 0\} x^8 = 0x^8$

[2] $\{1 + 0\} x^7 = x^7$

[3] $\{0 + 0 + 0\} x^6 = 0x^6$

[4] $\{1 + 0 + 0 + 0\} x^5 = x^5$

[5] $\{-4 + 0 + 0 + 7 + 0\} x^4 = 3x^4$

[6] $\{0 + 0 + 0 + 3\} x^3 = 3x^3$

[7] $\{ 0 + 7 + 0 \} x^2 = 7 x^2$

[8] $\{ -28 + 3 \} x = -25 x$

[9] $\{ 3 (-4) \} = -12$.

Ex. 8. Find $(x^4 + 2 x^3 + 5 x^2 + x + 1) (x^4 + x^3 + x^2 + x + 3)$.

Ans . We write as follows.

$$x^4 + 2 x^3 + 5 x^2 + x + 1$$

$$x^4 + x^3 + x^2 + x + 3$$

$$x^8 + 3 x^7 + 8 x^6 + 9 x^5 + 13 x^4 + 13 x^3 + 17 x^2 + 4 x + 3$$

Steps: -

[1] $\{ 1 (1) \} x^8 = x^8$

[2] $\{ 1 + 2 \} x^7 = 3 x^7$

[3] $\{ 1 + 2 + 5 \} x^6 = 8 x^6$

[4] $\{ 1 + 1 + 2 + 5 \} x^5 = 9 x^5$

[5] $\{ 3 + 2 + 2 + 1 + 5 \} x^4 = 13 x^4$

[6] $\{ 6 + 1 + 5 + 1 \} x^3 = 13 x^3$

[7] $\{ 15 + 1 + 1 \} x^2 = 17 x^2$

[8] $\{ 3 + 1 \} x = 4 x$

[9] $\{ 1 (3) \} = 3$.

8.2 Checking Method

Definition : - The **Beejank** of the polynomial is single digit obtained by repeated algebraic addition of all the co-efficients of the polynomial .

Examples

[1] The Beejank of the polynomial 'p' is denoted as B (p)

Thus B ($2x + 5$) = $2 + 5 = 7$.

B ($x + 2y + 3z + 1$) = $1 + 2 + 3 + 1 = 7$.

[2] When sum of the coefficients is negative or zero, we add ' 9 ' to find the

Beejank, Thus B ($x^3 - 4x^2 + 2x - 1$) = (- 2).

Adding 9 , we get B ($x^3 - 4x^2 + 2x - 1$) = $-2 + 9 = 7$

B ($24x^2 - 8y + 11$) = B ($2 + 4 - 8 + 1 + 1$ } = 0;

Adding 9 , we get B ($24x^2 - 8y + 11$) = $0 + 9 = 9$

Now we check the answers of the Ex.1, 2 and 3 by Beejank method and formula { 2 } above.

In Ex. [1] B ($3x + 2$) = 5 , B ($4x + 7$) = B (11) = 2

Consider Beejank of product of Beejank s we get

(5) (2) = 10 , and B (10) = $\underline{1}$

Similarly B ($12x^2+29x+14$)=B{1+2 + 2+9 + 1+4 } = $\underline{1}$.

As Beejanks are same the answer is correct.

In example [2] , $B\,\{(5x+1)(x+2)\}$

$$= B\,\{(5+1)(1+2)\}$$

$$= B\,(18) = \underline{9}$$

and $B\,(5x^2 + 11x + 2) = B\,(5+1+1+2) = \underline{9}$

As Beejanks are same the answer is correct

For Ex.3 , $B\,\{(5x + 3y + 2)(10x - 3y - 4)\} =$

$$B\,\{(5+3+2)(1-3-4)\} = B(-60) = B(-6) = 3$$

$$\text{And } B\,(50x^2 + 15xy - 9y^2 - 18y - 8) = B(5+6-9-9-8) = \underline{3}$$

As Beejanks are same the answer is correct

8.3 Multiplication And Addition / Subtraction. The multiplication, and addition or subtraction are simultaneously performed by applying Formula { 1 } for each pair followed by algebraic addition of coefficients of each pair as shown in following examples.

Ex. 9. Find $(6x+1)(2x+3) + (5x+4)(4x+1)$

Ans . We write the structure as follows.

$$\begin{array}{cccc} & 6x+1 & & 5x+4 \\ \times & & + & \\ & 2x+3 & & 4x+1 \\ \hline \multicolumn{4}{c}{32x^2 + 41x + 7} \\ \hline \end{array}$$

Steps: -

[1] $\{ (6)(2) + (5)(4) \} x^2 = 32 x^2$

[2] $\{ (6)(3) + (1)(2) \} x + \{(5)(1) + (4)(4) \} x = 41 x$

[3] $\{ (3)(1) + (4)(-1) \} = 7$

Checking : $B \{(6x+1)(2x+3)\} = B\{(7)(5)\} = 8$,

$B \{(5x+4)(4x+1)\} = B\{(9)(5)\} = 9$,

$B \{ (8) + (9) \} = B(17) = \underline{8}$

Now $B(32x^2 + 41x + 7) = B(3+2+4+1+7) = \underline{8}$

As Beejanks are same the answer is correct

Ex.10. Find $(2x+3)(6x+1)+(5x+3)(2x-1)-(x+1)(3x+7)$

Ans. We write the structure as follows.

$$\begin{array}{ccccccc} & 2x+3 & & 5x+3 & & x+1 \\ & & + & & - & \\ \times & 6x+1 & & 2x-1 & & 3x+7 \\ \hline & & & 19x^2+11x-7 & & \\ \hline \end{array}$$

Steps: -

[1] $\{ (2)(6) + (5)(2) - (1)(3) \} x^2 = 19 x^2$

[2] $\{ (2)(1) + (3)(6) \} x \{(5)(-1) + (3)(2) \} x + \{ (1)(7) + (1)(3) \} x = 11 x$

[3] $\{ (3)(1) +(3)(-1) - (1)(7) = -7$

Remark :- We can well realize that the VM methods are extremely simple as compared to current methods.

EXERCISE 8.1

Set A

Find the Beejank of the following polynomials :

[1] $3x + 2$

[2] $2x + 5$

[3] $4x - 7$

[4] $3x^2 + 4x - 2$

[5] $x^4 + 12x^2 - 3x + 8$

[6] $16x^3 - 21x^2 - 4$

[7] $(2x + 3)(4x - 1)$

[8] $(5x - 7)(2x + 3) - (3x + 4)(5x + 9)$

Set B

Multiply the following polynomials and check the answer by Beejank Method.

[1] $(2x - 1)(4x + 3)$

[2] $(2x + 5y)(3x - 4y)$

[3] $(7x - y)(y - 3x)$

[4] $(x^2 + 3x - 1)(x^2 - 2x - 3)$

[5] $(3x - y + 5)(x + y - 1)$

[6] $(x + y - z)(3x - y + 2z)$

[7] $(x^3 + 2x^2 + x - 2)(x^3 - x^2 + 3x - 1)$

[8] $(x + y + z + 1)(x - y + z + 1)$

[9] $(a^2 + 2b + 3c)(2a^2 + 4b - 7c)$

[10] $(x^3 - 2x^2 + 2x - 1)(x^2 + 4)$

[11] $(x^3 + 2x - 2)(x^4 - 2x^3 + 5x^2 + x - 1)$

[12] $(x^2 + 3x - 7/2)(2x^3 + 3x - 2)$

[13] $(x + 3y)(2x + 5y)$

[14] $(3x - 5y + 3)(5x + 1)$

[15] $(2x - y + 3)(x + y + 1)$

[16] $(6y - 8z + 4)(2x + z)$

[17] $(x + 2y - z + 4)(2y - x)$

[18] $(10x - 3y - 4)(5x + 3y + z)$

[19] $(4x + 5y)(3x - y)$

[20] $(x + 2y + 3z)(3x - y + 2z)$

[21] $(x - 2y + 3z - 4)(x - y + z + 3)$

[22] $(2x + y - 3z + 1)(3x - y + z - 1)$

[23] $(x + 2y)(x - 3y)$

[24] $(5 - x)(x - 3)$

[25] $(x^2 - 3x + 6)(x^2 + x + 4)$

Set C.

Find the polynomial in each case:

[1] $(x+2)(x-3)+(3x+1)(x-5)-(2x+3)(3x-2)$

[2] $(x+y)(x-2y)-(2x-y)(x+3y)+(4x+y)(y-3x)$

[3] $(x^2+2x-1)(x^2-x+1)+(x^2-5x+1)(2x^2+x-1)$

[4] $(x+2)(x+5)+(x+3)(x+4)$

[5] $(2x+5)(x+1)+(3x+2)(4x+7)$

[6] $(6x-1)(2x+3)-(x+3)(4x+2)$

[7] $(3x+4)(3-x)+(2x-1)(5x+1)$

[8] $(x^2+2)(3x^2-1)-(x^2+2)(2x^2+1)$

[9] $(x^3+x)(x+5)+(x^3+3)(x+2)$

[10] $(2x+1)(4x+3)+(x+5)(x+2)+(3x+2)(5x+1)$

[11] $(x+7)(x+5)-(x-1)(x+9)$

[12] $(7x+3)(3x-1)-(2x+5)(x-2)+(4x+3)(2x-1)$

[13] $(5x+3)(3x+5)-(2x+1)(3x+2)-(2x-1)(3-x)$

[14] $(4x+1)(x+5)+(3x+2)(2x+5)-(2x-1)(3x+4)+(4x-1)(2x+7)$

[15] $(x+2y)(3x-y)+(x+5)(2x-1)-(y-1)(y+3)$

[16] $(x^2+5x+1)(x^2-3x+2)+(x^2-2x+3)(x^2+4x-1)$

[17] $(3x^2+2x+2)(2x^2+x-7)-(6x^2+2x+11)(x^2-3x+1)$

ANSWERS 8.1

Set A [1] 5 [2] 7 [3] 6 [4] 5 [5] 9 [6] 9 [7] 6 [8] 9.

<u>Set B</u>

[1] $8x^2 + 2x - 3$

[2] $6x^2 + 7xy - 20y^2$

[3] $-21x^2 + 10xy - y^2$

[4] $x^4 + x^3 - 10x^2 - 7x + 3$

[5] $3x^2 + 2xy - y^2 + 2x + 6y - 5$

[6] $3x^2 + 2xy - xz - y^2 + 3yz - 2z^2$

[7] $x^6 + x^5 + 2x^4 + 2x^3 + 3x^2 - 7x + 2$

[8] $x^2 + 2xz - y^2 + z^2 + 2z + 2x + 1$

[9] $2a^4 + 8a^2b - a^2c + 8b^2 - 2bc - 21c^2$

[10] $x^5 - 2x^4 + 6x^3 - 9x^2 + 8x - 4$

[11] $x^7 - 2x^6 + 7x^5 - 5x^4 + 13x^3 - 8x^2 - 4x + 2$

[12] $2x^5 + 6x^4 - 4x^3 + 7x^2 - (33/2)x + 7$

[13] $2x^2 - 11xy + 15y^2$

[14] $15x^2 - 25xy + 18x - 5y + 3$

[15] $2x^2 + xy - y^2 + x + 4y - 3$

[16] $12xy - 8z^2 - 16xz + 6yz + 8x + 4z$.

[17] $-x^2 + xz + 4y^2 - 4x + 8y - 2yz$

[18] $50x^2 + 15xy - 9y^2 - 18y - 8$

[19] $12x^2 + 11xy - 5y^2$

[20] $3x^2 + 5xy - 2y^2 + 11xz + 6z^2 + yz$

[21] $x^2 - 3xy + 2y^2 + 4xz + 3z^2 + 5z - 2y - x - 12$

[22] $6x^2 + xy - y^2 - 7xz - 3z^2 + 4z - 2y + x - 1$

[23] $x^2 - xy - 6y^2$

[24] $-x^2 + 8x - 15$

[25] $x^5 - 3x^4 + 7x^3 + x^2 - 6x + 24$

Set C [1] $-2x^2 - 20x - 5$

[2] $-13x^2 - 5xy + 2y^2$

[3] $3x^4 - 8x^3 - 6x^2 + 9x - 2$

[4] $2x^2 + 14x + 22$

[5] $14x^2 + 36x + 19$

[6] $8x^2 + 2x - 9$

[7] $7x^2 + 2x + 11$

[8] $x^4 - x^2 - 7$

[9] $x^4 + 6x^3 + 3x^2 + 8x + 6$

[10] $24x^2 + 30x + 15$

[11] $4x + 44$

[12] $27x^2 + 3x + 4$

[13] $11x^2 + 20x + 16$

[14] $12x^2 + 55x + 4$

[15] $5x^2 + 5xy - y^2 + 9x + 2y - 8$

[16] $2x^4 + 4x^3 - 18x^2 + 21x - 1$

[17] $-13x^3 - 26x^2 + 19x + 3$

Division of Polynomials

8.4 In his book Swamiji has explained the method of division of polynomials. This method is, on many counts, similar to Horner's method of synthetic division. While the Horner's method is restricted to divisor of the type (x + a) or (a x + b), Swamiji explains the method of division of any n^{th} degree polynomial by any m^{th} degree polynomial where n > m.. This can be considered as one of the best contribution of Swamiji to mathematics.

But this method is restricted to polynomials in only one variable. We have extended this method to polynomials in two or three variables.

Sutra परावर्त्य योजयेत् Paravartya yojayet

(Transpose and apply)

8.5 When the dividend is n^{th} degree polynomial and divisor is m^{th} degree polynomial in x such that m < n.

Method :- Let the dividend is n^{th} degree polynomial

$b_0 x^n + b_1 x^{n-1} + b_2 x^{n-2} + \text{———————} + b_n$ and the divisor is $b0 \neq 0$

$a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \text{———————} + a_m$, such that m < n. and $a0 \neq 0$

Note that there are (n+1) terms in dividend,

and (m+1) terms in divisor.

Steps

[1] We arrange the dividend and divisor in descending powers of x, by inserting the term $(0.x^p)$, p < m < n, if the

term in x^p is absent.

[2] The digits, of the divisor as modified by Sutra are

$-a_1/a_0 = u_1, -a_2/a_0 = u_2, -a_3/a_0 = u_3, -a_m/a_0 = u_m$.

We call them as modified divisor (M. D.) Write the M. D. in the chart as shown.

[3] The method of division and placement of digits is similar to method of synthetic division.

M.D.	x^n	x^{n-1}	x^{n-2} ----------------			x^0
u_1	b_0	b_1	b_2			b_n
u_2		c_1	c_2			c_m
			d_1			d_m
u_m						
	b_0	k_1	k_2	k_{n-m}	k_m	k_n

Steps:

[1] Write $b_0 = k_0$ as it is in the answer line.

[2] Write $u_1 b_0 = c_1$, $u_2 b_0 = c_2$, $u_3 b_0 = c_3$, ————, $u_m b_0 = c_m$, as shown.

[3] Find $k_1 = b_1 + c_1$.

[4] Write $u_1 k_1 = d_1$, $u_2 k_1 = d_2$, $u_3 k_1 = d_3$ ————, $u_m k_1 = d_m$, as shown.

[5] Find $k_2 = b_2 + c_2 + d_1$.

[6] The procedure is continued up to (n +1) th column.

Out of total (n+1) digits in the answer line first (n+1-m) digits are coefficients of quotient polynomial of x^{n-m} degree and next m digits are coefficients of remainder

polynomial of x $^{m-1}$ degree.

[7] Quotient $Q = (1/a_0)\{k_0 x^{n-m} + k_1 x^{n-m-1} + \text{———} + k_{n-m}\}$.

[8] Remainder $R = \{k_m x^{m-1} + k_{m+1} x^{m-2} + \text{———} + k_n\}$

Ex 11. Divide $x^7 - 2x^6 + 7x^5 - 5x^4 + 13x^3 - 8x^2 - 4x + 2$

by $x^4 - 2x^3 + 5x^2 + x - 1$.

Ans. Here n = 7 , m = 4 , a_0 = 1. write the table as follows.

M. D.	x^7	x^6	x^5	x^4	x^3	x^2	x	x^0
	1	-2	7	-5	13	-8	-4	2
2		2	-5	-1	1			
-5			0	0	0	0		
-1				4	-10	-2	2	
1					- 4	10	2	-2
	1	0	2	- 2	0	0	0	0

Steps:

[1] Write 1 as it is in the answer line.

[2] Write (1) (2) = 2 , (1) (-5) = -5 , (1) (-1) = -1 , (1) (1) = 1 as shown.

[3] Write (-2) + (2) = 0, in next column of answer line

[4] Write (0) (2) = 0 , (0) (-5) = 0 , (0) (-1) = 0 , (0) (1) = 0 , as shown.

[5] Write (7) + (-5) + (0) = 2, in next column of answer line

[6] Write (2) (2) = 4 , (2) (-5) = -10 , (2) (-1) = -2 , (2) (1) = 2, as shown.

[7] Write (-5) + (-1) + (0) +(4) = - 2, in next column of answer line.

[8] Write (-2) (2) = -4,(-2)(-5)=10 ,(-2) (-1) = 2,(-2)(1)=-2, as shown.

[9] Write (13) + (1) + (0) + (-10) + (- 4) = 0, in next column of answer line.

[10] Write (-8) + (0) + (-2) +(10) = 0, in next column of answer line.

[11] Write (-4) + (2) + (2) = 0, in next column of answer line

[12] Write (2) + (-2) = 0, in next column of answer line

Hence Q = (1 / 1) $\{x^3 + 0x^2 + 2x - 2\} = x^3 + 2x - 2$.

$R = 0x^3 + 0x^2 + 0x + 0 = 0.$

Checking Method :

We use remainder theorem and beejank to check the answer of the division

Steps:

[1] Find B (Dividend) = x (say) where B = Beejank.

[2] Find B (Divisor)

[3] Find B (Quotient)

[4] Find B (Remainder)

[5] Find B{B(Quotient) x B (Divisor)+B(Remainder)}=y(say)

[6] If x = y then answer is correct.

Now we check the answer of Ex. 11 by above method.

Steps:

[1] B (Dividend) = $B(x^7 - 2x^6 + 7x^5 - 5x^4 + 13x^3 - 8x^2 - 4x + 2$

$= B(1 - 2 + 7 - 5 + 13 - 8 - 4 + 2) = 4.$

[2] B (Divisor) $= B(x^4 - 2x^3 + 5x^2 + x - 1) = B$

$(1 - 2 + 5 + 1 - 1) = 4.$

[3] B (Quotient) = $B(x^3 + 2x - 2) = B(1 + 2 - 2) = 1$

[4] B (Remainder) = B (0) = 0

[5] B {B (1) x B (4) + B (0)} = 4

[6] Here Beejank in step 1 and step 5 are equal hence answer is correct.

Ex. 12 Divide $9x^4 - x^2 + 11x - 21$ by $3x^2 + x - 5$.

Ans :- Here n - 4 , m = 2 , $a_0 = 3$. The term $0x^3$ is inserted in dividend.

The table of various operations is as follows.

(108)

M. D.	x^4	x^3	x^2	x	c
- 1 / 3	9	0	-1	11	-21
5 / 3		-3	15		
			1	-5	
				-5	25
	9	-3	15	1	4

[1] Write 9 as it is in the answer line.

[2] Write (9) (-1/3) = -3 , (9) (5/3) = 15 as shown.

[3] Write (0) + (-3) = -3, in next column of answer line.

[4] Write (-3) (-1/3) = 1 , (-3) (5/3) = -5 as shown.

[5] Write (-1) + (15) + (1) = 15 in next column of answer line.

[6] Write (15) (-1/3) = -5 , (15) (5/3) = 25 as shown.

[7] Write (11) + (-5) + (-5) = 1, in next column of answer line.

[8] Write (-21) + (25) = 4 in next column of answer line.

Hence $Q = (1/3)(9x^2 - 3x + 15) = 3x^2 - x + 5$.

$R = (1)x + 4 = x + 4$.

We check the answer by Beejank method

Steps:

[1] B (Dividend) = B $(9x^4 - x^2 + 11x - 21)$ = B (-2) = 7

[2] B (Divisor) = B $(3x^2 + x - 5)$ = B (-1) = 8

[3] B (Quotient) = B $(3x^2 - x + 5)$ = B (7) = 7

[4] B (Remainder) = B $(x + 4)$ = 5

[5] B {B (Quotient) x B (Divisor) + B (Remainder)} { (7 x 8) + 5 } = B (61) = 7.

[6] Here Beejank in step 1 and step 5 are equal hence answer is correct.

Ex. 13. Divide $x^5 + 3x^4 - 2x^3 - 8x$ by $2x^3 + 3x - 2$.

Ans :- Here $n = 5$, $m = 3$, $a_0 = 2$. The term $0x^2$ and $0x^0$ are inserted in dividend and term $0x^2$ is inserted in divisor.

M. D.	x^5	x^4	x^3	x^2	x	c
0	1	3	-2	0	-8	0
- 3 / 2		0	-3/2	1		
1			0	-9 /2	3	
				0	21/4	-7 /2
	1	3	-7 / 2	-7 /2	1/4	-7 / 2

Hence $Q = (1/2)(x^2 + 3x - 7/2)$

$R = (-7/2)x^2 + (1/4)x - 7/2$

<u>Note</u> :

1] Do not multiply remainder terms by $(1/a_0)$.

2] Apply remainder theorem for checking the answer by Beejank method .

8.6 When dividend is second degree homogeneous polynomial in x and y, and divisor is first degree polynomial in x and y.

Method :- Let the dividend is $a x^2 + h x y + b y^2$ and divisor is $m x + n y$.

Steps

[1] The coefficients of dividend are arranged as shown.

[2] The modified divisor by sutra is $(-n/m) = d$ (say).

[3] The table of various operations is as follows.

	x^2	xy	y^2
	a	h	b
M.D.			
d		a d	q d
	a	q	r

Steps:

[1] Write 'a' as it is in first column of answer line.

[2] Write $(a)(d) = a d$ as shown.

[3] Write $h + a d = q$ in the next column.

[4] Write $(q)(d) = q d$ as shown.

[5] Write $b + q d = r$ in the next column

Answer: Quotient $Q = (1/m)\{ a x + q\ y)$ and

Remainder $R = r\ y^2$

Ex. 14. Divide $2x^2 + 11xy + 15y^2$ by $2x + 5y$

	x^2	xy	y^2
	2	11	15
M.D.			
- 5 / 2		-5	- 15
	2	6	0

Steps:

[1] Write 2 as it is in first column of answer line.

[2] Write (2) (-5/2) = -5 as shown.

[3] Write 11 + (-5) = 6 in the next column.

[4] Write (6) (-5/2) = -15 as shown.

[5] Write 15 + (- 15) = 0 in the next column.

Answer : Quotient Q = (1 / 2) { 2 x + 6 y) = x + 3 y.

Remainder R = $0y^2 = 0$.

Checking :

Steps:

[1] B $(2x^2 + 11xy + 15)$ = B (10) = 1.

[2] B (2 x + 5 y.) = 7

[3] B (x + 3 y) = 4

[4] B (0) = 0

[5] B {(7 x 4) + 0} = B (28) = 1

AS Beejank in step 1 and step 5 are equal answer is correct.

8.7 When dividend is second degree polynomial in x and y and divisor is first degree polynomial in x and y.

Method :- Let the dividend be $a x^2 + h x y + b y^2 + g x + f y + c$ and divisor is $m x + n y + p$.

The table of various operations is as follows.

	x^2	xy	x	y^2	y	cons.
	a	h	g	b	f	c
M.D.						
d_1		ad_1	ad_2			
d_2				k_1d_1	k_1d_2	
					k_2d_1	k_2d_2
	a	k_1	k_2	r_1	r_2	r_3

Steps:

[1] The coefficients of dividend are arranged as shown.

[2] The modified divisor by sutra is:$(- n/m) = d_1$,$(- p / m) = d_2$

[3] Write 'a' as it is in first column of answer line.

[4] Write $(a)(d_1) = ad_1$ and $(a)(d_2) = ad_2$ as shown.

[5] Write $h + ad_1 = k_1$, $g + ad_2 = k_2$ in next column.

[6] Write k_1d_1 and k_1d_2 as shown.

[7] Write k_2d_1 and k_2d_2 as shown.

[8] Write $b + k_1d_1 = r_1$, $f + k_1d_2 + k_2d_1 = r_2$, and $c + k_2d_2 = r_3$.

Answer: Quotient $Q = (1 / m) \{ a x + k_1y + k_2)$.

Remainder $R = r_1 y^2 + r_2 y + r_3$.

Note :- Out of the six digits in the answer line, quotient consists of digits equal to number of terms in the divisor i.e.3

8.8 When dividend is second degree homogeneous polynomial in x y and z , and divisor is first-degree homogeneous polynomial in x and y and z .

Method :- Let the dividend be $a x^2 + h x y + b y^2 + g x z + f y z + c z^2$

and divisor is $m x + n y + p z$.

The table of various operations is as follows.

	x^2	$x y$	$x z$	y^2	$y z$	z^2.
	a	h	g	b	f	c
M.D.						
d_1		ad_1	ad_2			
d_2				k_1d_1	k_1d_2	
					k_2d_1	k_2d_2
	a	k_1	k_2	r_1	r_2	r_3

Steps

[1] The coefficients of dividend are arranged as shown.

[2] The modified divisor by sutra is :$(- n/m = d_1$, $(- p / m) = d_2$

Steps [3] to [8] are exactly same as steps in 8.7 above.

Answer: Quotient $Q = (1 / m) \{ a x + k_1 y + k_2 z)$.

Remainder $R = r_1 y^2 + r_2 y z + r_3 z$.

Note:- Out of the six digits in the answer line, quotient consists

of digits equal to number of terms in the divisor i.e.3.

Ex. 15. Divide $15x^2 - 25xy + 18x - 5y^2 + 6$ by $5x + 1$.

[Note that term in y is absent in dividend and divisor]

	x^2	xy	x	y^2	y	cons.
	15	-25	18	-5	0	6
M.D.						
0		0	-3			
-1/5				0	5	
					0	-3
	15	-25	15	-5	5	3

Steps:

[1] The coefficients of dividend are arranged as shown.

[2] The modified divisor by sutra is: (-0/5)=0,(-1/5) = -1/5.

[3] Write 15 as it is in first column of answer line.

[4] Write (15) (0) = 0 and (15) (-1/5) = -3 as shown.

[5] Write (-25) + 0 = -25 , 18 + (-3) = 15 in next column.

[6] Write (-25) (0) = 0 and (-25) (-1/5) = 5 as shown.

[7] Write (15) (0) = 0 and (15) (-1/5) = -3 as shown.

[8] Write (-5) + 0 = -5 , 0 + 5 + 0 = 5 , and 6 + (-3) = 3.

Answer: Quotient Q = $(1/5)\{15x^2-25y+15\}=3x^2-5y+3$

Remainder R = $-5y^2 + 5y + 3$.

Checking :

Steps

[1] B $(15x^2 - 25xy + 18x - 5y^2 + 6)$ = B (6-7+9-5+6) = 9

[2] B $(5x + 1) = 6$

[3] B $(3x^2 - 5y + 3) = 1$

[4] B $(-5y^2 + 5y + 3) = 3$

[5] B {(6) (1) +3 } = 9

[6] As Beejank in step 1 and 5 are equal the answer is correct.

Ex. 16. Divide $-2x^2 + 11xy + 21y^2 - 10z^2 - 12xz - yz$ by $2x + 3y + 2z$.

	x^2	xy	xz	y^2	yz	z^2.
	-2	11	-12	21	-1	-10
M.D.						
-3 / 2		3	2			
-1				-21	-14	
					15	10
	-2	14	-10	0	0	0

Steps:

[1] The coefficients of dividend are arranged as shown.

[2] The modified divisor by sutra is: (-3/2) = -3/2, (-2/2) = -1.

[3] Write -2 as it is in first column of answer line.

[4] Write (-2) (-3/2) = 3 and (-2) (-1) = 2 as shown.

[5] Write (11) + 3 = 14 , -12 + (2) = -10 in next column.

[6] Write (14) (-3/2) = -21 and (14) (-1) = -14 as shown.

[7] Write (-10) (-3/2) = 15 and (-10) (-1) = 10 as shown.

[8] Write 21 + (-21) = 0, (-1) + (-14)+15=0, and (-10) + 10 = 0.
Answer: Quotient Q = (1 / 2) { -2 x + 14 y -10 z)

$$= -x + 7y - 5z$$

and Remainder R = $0y^2 + 0y + 0 = 0$.

8.9 When dividend is second degree polynomial in x, y and z, and divisor is first degree polynomial in x, y and z.

Method :- Let the dividend be

$ax^2 + hxy + by^2 + gyz + fxz + cz^2 + px + qy + s$

and divisor is $wx + my + nz + k$.

Steps

[1] The coefficients of dividend are arranged as shown.

[2] The modified divisor by sutra is $(-m/w) = d_1$,

$(-n/w) = d_2$ and $(-k/w) = d_3$ (say).

[3] The table of various operations is as follows.

	x^2	xy	xz	x	y^2	yz	y	z^2	z	con
	a	h	f	p	b	g	q	c	r	s
M.D.										
d_1		ad_1	ad_2	ad_3						
d_2					k_1d_1	k_1d_2	k_1d_3			
d_3					k_2d_1	——	k_2d_2	k_2d_3		
						k_3d_1	——	k_3d_2	k_3d3	
	a	k_1	k_2	k_3	r_1	r_2	r_3	r_4	r_5	r_6

Where $k_1 = h + a\,d_1$, $k_2 = f + ad_2$ etc.

Answer: $Q = (1/w)\{ax + k_1 y + k_2 z + k_3)$ and

$$R = r_1 y^2 + r_2 yz + r_3 y + r_4 z^2 + r_5 z + r_6$$

Ex. 17. Divide $3x^2 + 2y^2 + z^2 - 5xy + 2xz + 4yz + 3x - 2y + 3$ by $3x - y + 2z - 1$.

	x^2	xy	xz	x	y^2	yz	y	z^2	z	con
M.D.										
	3	-5	2	3	2	4	-2	1	0	3
1/3		1	-2	1						
-2/3					-4/3	8/3	-4/3			
1/3						0	....	0	0	
							4/3	....	-8/3	4/3
Ans.	3	-4	0	4	2/3	20/3	-2	1	-8/3	13/3

$Q = (1/3)(3x^2 - 4xy + 4x)$ and

$R = (2/3)y^2 + (20/3)yz - 2y + z^2 - (8/3)z + (13/3).$

Note the order in which the terms of dividend are arranged.

Checking:

Steps

[1] $B(3x^2 + 2y^2 + z^2 - 5xy + 2xz + 4yz + 3x - 2y + 3) = \mathbf{2}$

[2] $B(3x - y + 2z - 1) = 3$

[3] $B(1/3)(3x^2 - 4xy + 4x) = (1/3)(3) = 1$

[4] $B(2/3)y^2 + (20/3)yz - 2y + z^2 - (8/3)z + (13/3) = 8.$

[5] $B\{(3)(1) + 8\} = B(11) = 2.$

[6] As Beejank in step 1 and step 5 are equal the is correct.

8.10 Special Case

Ex. 13. Divide $3xy + 2y^2 + 7x + 5y + 2$ by $3x + 2y - 1$.

Answer :— This is a case where term in x^2 is absent.

Note the arrangement of the dividend and corresponding divisor

$2y + 3x - 1$	y^2	yx	y	x^2	x	c
$-3/2$	2	3	5	0	7	2
$1/2$		-3	1			
				0	0	
					-9	3
	2	0	6	0	-2	5

Thus $Q = (1/2)\, 2y + 0x + 6 = y + 3$

$R = 0x^2 - 2x + 5 = -2x + 5$.

Ex. 14. Divide $4xy + 14xz - 10yz - 23z^2$ by $2y + 7z$

Ans : - Note that term in ' x ' is absent in the divisor. Hence we arrange the dividend and divisor as shown

$2y + 7z + 0x$	y^2	yz	yx	z^2	zx	x^2
$-7/2$	0	-10	4	-23	14	0
0		0	0			
				35	0	
					-14	0
	0	-10	4		12	0 0

Hence $Q = (1/2)\{0y^2 - 10z + 4x\} = -5z + 2x$

$R = 12z^2$.

Exercise 8.2

SET A Divide

[1] $6x^2 - 11x - 10$ by $2x - 5$

[2] $3x^2 - 17x + 10$ by $5 - x$

[3] $2x^4 - 3x^3 + 5x^2 + x - 9$ by $x + 1$

[4] $3x^3 + 22x^2 + x + 3$ by $3x + 1$

[5] $x^5 - 3x^4 + 7x^3 + x^2 - 6x + 24$ by $x^3 + x + 4$.

[6] $16x^3 - 2x^2 + 5x - 3$ by $4x^2 + 1$

[7] $x^4 + 2x^3 - x^2 - 3x + 1$ by $2x^2 + 3x - 1$

[8] $x^4 + 2x^3 + 7x^2 - 5x + 4$ by $x^2 + 2x - 1$

[9] $x^8 + 7x^6 + 5x^5 + 9x^3 + 3x^2 + 12x - 7$ by $x^3 - 1$

[10] $x^9 - 5x^8 + 7x^7 - x^6 - 7x^5 - 3x^4 - 3x^3 + x^2 + 5x - 2$
by $x^6 + 2x^5 - x^3 - 2x + 1$.

[11] $2x^5 + 6x^4 - 4x^3 + 7x^2 - (33/2)x + 7$ by $x^2 + 3x - (7/2)$.

[12] $12x^2 + 11x - 5$ by $3x - 1$.

[13] $x^5 - 3x^4 + 7x^3 + 2x^2 + 6x + 30$ by $x^2 - 3x + 6$.

[14] $-x^2 + 8x + 10$ by $5 - x$.

[15] $x^4 - 12x^2 + 37$ by $x^2 + 6$.

[16] $x^8 - 3x^7 + 4x^6 + 2x^5 - 2x^4 + 3x^3 + 5x^2 - x - 12$
by $x^3 - x^2 + x + 3$.

[17] $x^7 + x^6 - x^5 - x^4 - 5x^3 + 5x^2 - 6$ by $x^3 + x^2 - x + 2$

[18] $-2x^2 + 11x^2 + 21x^2 - 11x^2 - 12x - 18$ by $2x^2 + 3x + 2$.

[19] $x^5 + x^4 + x^3 + x^2 + x + 1$ by $2x^2 + x$

[20] $x^7 + x^5 + 3x^4 + 3x^3 + 7x^2 - 25x - 12$ by $x^4 + 7x^2 + 3$

<u>SET B</u> <u>Divide</u>

[1] $2x^2 + xy - y^2 + x + 4y - 3$ by $x + y + -1$

[2] $8x^2 + 5y^2 + 16xy + x + y - 1$ by $4x + 2y - 1$

[3] $6x^2 + 3xy - 3z^2 - yz + 2xz$ by $3x - z$.

[4] $3x^2 + 5xy + 11xz - yz + 2y^2 + 7z^2$ by $x + 2y + 3z$.

[5] $3x^2 + 7xy + 2y^2 + 7yz + 6z^2 + 11xz + 6x + 8y + 16z + 8$

by $x + 2y + 3z + 2$.

[6] $-x^2 + xz - 4x + 4y^2 - 3z^2 + 8z$ by $x + 2y - z + 4$.

[7] $12xy - 16xz + 8x - 8z^2 + 4z + 6yz$ by $6y - 8z + 4$.

[8] $x^2 - 2xy + 7y^2$ by $x + 4y$.

[9] $4x^2 + 8xy + 3y^2$ by $2x + y$.

[10] $3x^2 + 5xy + 2y^2$ by $4x - 1$

[11] $6x^2 + 3xy + y^2 + 5x + y - 1$ by $2x + y - 1$.

[12] $x^2 + 3xy - 2xz + 4y^2 + 2yz - 5z^2$ by $x + y - 2z$.

[13] $4x^2 + 4xy + 3xz - y^2 + 5yz + (1/2)z^2$ by $2x - 2y + z$.

[14] $12x^2 + xy - 8xz + 2y^2 + 20yz - 34z^2$ by $4x + 3y - 8z$.

[15] $x^2 + 2xy - 3xz - 3y^2 + 4yz + 2z^2 + 4x + 2y - 6z + 4$

by $x - y - z + 2$.

[16] $2x^2 + 9xy + 3xz + 7y^2 + 2yz + z^2 + 2x - 2z - 18$

by $2x + 4y + z - 6$.

Answers 8.2

<u>SET A</u>

[1] $Q = 3x = 2$	$R = 0$
[2] $Q = -3x - 2$	$R = 20$
[3] $Q = 2x^3 - 5x^2 + 10x - 9$,	$R = 0$
[4] $Q = x^2 + 7x - 2$	$R = 5$
[5] $Q = x^2 - 3x + 6$	$R = 0$
[6] $Q = 4x - (1/2)$	$R = x - (5/2)$
[7] $Q = (1/2)x^2 (1/4)x - (5/8)$	$R = (-7/8)x + (3/8)$
[8] $Q = x^2 + 8$	$R = -21x + 12$
[9] $Q = x^5 + 7x^3 + 6x^2 + 16$	$R = 9x^2 + 12x + 9$
[10] $Q = x^3 + 3x^2 + x - 2$	$R = 0$
[11] $Q = x^3 + 3x - 2$	$R = 0$
[12] $Q = 4x + 5$	$R = 0$
[13] $Q = x^3 + x + 5$	$R = 15x$
[14] $-x + 3$.	$R = -5$
[15] $Q = x^2 - 18$	$R = 145$
[16] $Q = x^3 - 2x^2 + 3x - 4$	$R = 0$
[17] $Q = x^4 - 3x - 2$	$R = 4x^2 + 4x - 2$
[18] $Q = -x^3 + 7x^2 + x - 14$	$R = 28x + 10$

[19] $Q = (1/2)x^3 + (1/4)x^2 + (3/8)x + (5/16)$. $R = (11/16)x + 1$

[20] $Q = x^3 + x - 4$ $R = 0$

SET B

[1] $Q = 2x - y + 3$ $R = 0$

[2] $Q = 2x + 3y + (3/4)$ $R = -y^2 + (5/2)y - ($
$1/4)$

[3] $Q = 2x + y + (4/3)z$ $R = (-5/3)z^2$

[4] $Q = 3x - y + 2z$ $R = 14y^2 - 2yz + z^2$

[5] $Q = 3x + y + 2z$ $R = 6y + 12z + 8$

[6] $Q = -x + 2y$ $R=2yz-8y-3z^2 + 8z$

[7] $Q = z + 2x$ $R = -5z^2 - 10zx$

[8] $Q = x - 6y$ $R = 31y^2$

[9] $Q = 2x + 3y$ $R = 0$

[10] $Q = (3/4)x + (23/16)y$ $R = (55/16)y^2$

[11] $Q = 3x + 4$ $R = y^2 - 3y + 3$

[12] $Q = x + 2y$ $R = 2y^2 + 6yz - 5z^2$

[13] $Q = 2x + 4y + (1/2)z$ $R = 7y^2 + 2yz$

[14] $Q = 3x - 2y + 4z$ $R = 8y^2 + 8yz - 2z^2$

[15] $Q = x + 3y - 2z + 2$ $R = 0$

[16] $Q = x + (5/2)y + z + 4$ $R = -3y^2 - (9/2)yz - y + 6$

❖ ❖ ❖

CHAPTER 9
FACTORIZATION OF POLYNOMIALS

9.1 First we shall study the method of factorization of quadratic polynomials in one variable, suggested by Swamiji in his book. This method, though similar to current methods, is much quicker and it gives roots of the equation without evaluating the factors.

Secondly, we shall study the Vedic method for factorization of homogeneous and non-homogeneous polynomials in two or three variables and solution of cubic equations. Here we tried to justify the Vedic method by considering its limitations.

SUTRAS

(1) आनुरूप्येण (Anurupyena)
(Proportionately)

(2) आद्यमाद्येनामन्त्यमन्त्येन (Adyamadyenamantyamantyena)
(First by the first, and last by the last)

(3) लोपनस्थापनाभ्याम् (Lopansthapanabhyam)
(By alternate elimination and retention)

9.2 Factorization of quadratic polynomial in one variable

Steps :-

(1) We write the polynomial as $a x^2 + bx + c$

(2) We split 'b' in two parts such that $a:u = v:c$ where $b = u+v$

(3) Let ratio in the simple form is $p:q$. i.e. $a/u = v/c = p/q$

(4) Then first factor is :-- $px + q$.

(5) For second factor, by Sutra (2) we get $(a/p) = k$, $(c/q) = m$ (say),

(6) Then second factor is :- $kx + m$.

Justification :- Consider the product

$$(px + q)(kx + m) = pkx^2 + (pm + qk)x + qm.$$

$$ax^2 + [(pc/q) + (qa/p)]x + c$$

$$ax^2 + [v + u]x + c$$

$$ax^2 + bx + c$$

Thus the factors of $ax^2 + bx + c$ are $(px + q)(kx + m)$.

9.3 Roots of the equation $ax^2 + bx + c = 0$

The factors of the LHS. of the equation are $(px + q)(kx + m)$

Hence roots of the equation are : $x = (-q/p)$ and $x = (-m/k)$

i.e. $x = (-u/a)$ and $x = (-cp/aq) = -(c/a)(v/c)$

$= (-v/a)$.

The roots of the equation are $[-u/a, -v/a]$.

Remark :- The roots can be evaluated without finding actual factors.

Note :

(1) This method is applicable only when $D = b^2 - 4ac$ is a perfect square number. In this case the equation $ax^2 + bx + c = 0$ has rational roots.

(2) If $D > 0$ but not perfect square, then equation has irrational roots.

(3) If $D < 0$ then equation has complex roots.

Ex. 1:- Factorize $x^2 + 6x + 8$ and find the roots of the equation $x^2 + 6x + 8 = 0$.

Ans :- Here $a = 1$, $b = 6$, and $c = 8$. we find that $6 = 4 + 2$ such that $1 : 4 = 2 : 8$, hence $u = 4$ and $v = 2$.

Let ratio in the simple form is $1 : 4$

Hence first factor is :- $(x + 4)$

For second factor: $k = (1/1) = 1$, $m - (8/4) = 2$,

Hence second factor is :- $(x + 2)$

The roots of the equation are : $-u/a = -4/1 = -4$,

$-v/a = -2/1 = -2$.

Ex. 2 :- Factorize $12x^2 - 17x - 7$ and find its roots

Here $a = 12$, $b = -17$, $c = -7$. We find that $-17 = 4-21$ such that $12:4 = -21 : -7$. Hence $u = 4$ and $v = -21$.

The ratio in the simple form is. $3 : 1$

Hence first factor is :- $(3x + 1)$

For second factor $k = 12/3 = 4$, and $m = -7/1 = -7$

Hence second factor is :- $(4x - 7)$.

The roots of the equation are $x = -u/a = -4/12 = -1/3$

$x = -v/a = -(-21/12) = 7/4$.

9.4 General method for roots of the equation

$ax^2 + bx + c = 0$, $a \neq 0$.

We know that for any value of D the roots are given by

$x = (1/2a) [-b \pm (D)]$, Where $D = b^2 - 4ac$

We rewrite above equation as $2 ax + b = + (D)$ or

$$d/dx [ax^2 + bx + c] = + (D)$$

Above equations gives roots ofthe equation for allvalues of D.

Ex. 3 :- Find the roots of $6x^2 - 11x - 10 = 0$

Ans :- Hear a = 6, b=-11, c=-10. Hence D=121+240= 361

Thus roots are given by :- $2(6)x+(-11) = 19$, or -19

i.e. $12x - 11 = 19$ gives $x = 5/2$,

and $12x - 11= -19$ gives $x = -2/3$.

The roots are [5/2, -2/3].

Ex. 4 :- Find the roots of $x^2 - 2x - 1 = 0$.

Here a = 1, b = -2, c = -1. Hence D = 4 + 4 = 8.

Thus roots are given by :- $2x-2=2(2)^{1/2}$ or $-2(2)^{1/2}$.

i.e. $x - 1 = (2)$ gives $x = 1 + \sqrt{2}$

and $x - 1 = (2)$ gives $x = 1 - \sqrt{2}$.

Ex. 5 :- Find the roots of $x^2 - 10x + 29 = 0$.

Here a=1, b = -10, c = 29. Hence D = 100-116=-16

The roots are given by :- $2x - 10 = \sqrt{(-16)} = 4 i$

Where $i^2 = -1$.

i.e. $x - 5 = 2 i$ gives $x = 5 + 2 i$

and $x - 5 = - 2 i$ gives $x = 5 - 2 i$

EXERCISE

Set A :- Factorize the following polynomials :-

1] $x^2 + 8x = 15$ 2] $8x^2 + 30x + 7$

3] $3x^2 - 17x + 10$ 4] $6x^2 - 13x - 5$

5] $-12x^2 - 10x + 50$ 6] $-x^2 + 8x - 15$

Set B :- Find the roots of the following equations :-

1] $12x^2 + 11x - 5 = 0$ 2] $-3x^2 - 5x + 12 = 0$

3] $3x^2 - 17x + 10 = 0$ 4] $6x^2 + 19x + 10 = 0$

5] $x^2 - 2x - 17 = 0$ 6] $4x^2 + 12x - 11 = 0$

7] $x^2 - 4x + 13 = 0$ 8] $x^2 - 12x + 37 = 0$

ANSWERS

Set A :-

[1] $(x + 3) (x + 5)$ [2] $(2x + 7) (4x + 1)$ {3}$(3x - 2) (x -3)$

[4] $(2x - 5)(3x + 1)$ [5] $(-6x + 10)(2x + 5)$ [6] $(5 - x)(x - 3)$

Set B :- [1] $x = 1/3, -5/4$ [2] $x = -3, 4/3$

[3] $x = 2/3, 5$ [4] $x = -2/3, -5/2$

[5] $x = -1 + (2)$ [6] $x = -3/2 + (5)$

[7] $x = 2 + 3i$ [8] $x = 6 + i$

9.5 Factorization of second degree homogeneous polynomial in two variables.

Steps :-

[1] We write the polynomial as $ax^2 + 2hxy + by^2$

[2] We split '2h' in two parts such that $a:u = v:c$ where $2h = u+v$.

[3] Let the ratio in the simple form is $p:q$. i.e. $a/u = v/c = p/q$

[4] Then first factor is $px + qy$.

[5] For second factor, by Sutra (2) we get $(a/p)=k$, $(c/q)=m$ (say),

[6] Then second factor is $kx + my$.

Note : [1] The polynomial $ax^2 + 2hxy + by^2$ is factoriazable into rational factors only when $h^2 - ab > 0$.

[2] This factorization helps us in finding equations of pair of lines through origin represented by the equation $ax^2 + 2hxy + by^2=0$.

Ex. 6 :- Factorize $x^2 - xy - 6y^2$.

Ans :- Here $a = 1$, $2h = -1$, $b = -6$.

We find that $-1 = 2 - 3$ such that $1:2 = -3 : -6$.

The ratio in the simple form is : $1 : 2$

Hence first factor is $(x + 2y)$.

For second factor $k = 1/1 = 1$, and $m = -6/2 = -3$

Hence second factor is $(x - 3y)$.

EXERCISE Factorize :

[1] $4x^2 - 3xy - y^2$ [2] $6x^2 + 19xy + 10y^2$

[3] $2a^2 - 3ab - 2b^2$ [4] $2x^2 - 13xy - 7y^2$

[5] $12x^2 + 11xy - 5y^2$

ANSWERS

[1] $(x - y)(4x + y)$ [2] $(2x + 5y)(3x + 2y)$

[3] $(a - 2b)(2a + b)$ [4] $(x - 7y)(2x + y)$

[5] $(4x + 5y)(3x - y)$

9.6 Factorization of second degree non homogeneous polynomial in two variables

Steps :-

[1] We write the polynomial as
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ ------------ (A)

[2] Put $y = 0$ in (A), we get $ax^2 + 2gx + c$ (Elimination)
Let factors are $(px + r)$ and $(kx + n)$.

[3] put $x = 0$ in (A), we get $by^2 + 2fy + c$ (Elimination)
Then factors are :- $(qy + r)$ and $(my + n)$
(Note the argument)

[4] We set the cyclic arrangement of the factors as
$(px + r)$ $(qy + r)$, $(kx + n)$ $(my + n)$

[5] Hence factors of the polynomial (A) are
$(px + qy + r)$ $(kx + my + n)$ (Ratention)
(Note the argument)

Note :

[1] The polynomial (A) is factoriazable only when

$$d^1 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

[2] When $D^1 \neq 0$, either polynomials in steps 2) and 3) are not factoriazable or we may get incorrect factors because this method works irrespective of value of 'h', hence the correctness of the factors is to be decided by checking methods or actual multiplication. (Ref. Ex. 8 and 9).

[3] This factorization helps us in finding equations of pair of lines reqresented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Ex. 7 :- Factorize $2x^2 + xy - y^2 + x + 4y - 3$. Given $D^1 = 0$

Steps :-

[1] We write the polynomial as $2x^2 + xy - y^2 + x + 4y - 3$. --- (A)

[2] Put $y = 0$ in (A), we get $2x^2 + x - 3$.
The factors are :- $(x - 1)$ and $(2x + 3)$

[3] Put $x = 0$ in (A), we get $-y^2 + 4y + -3$ (Elimination)
Then factors are :- $(y - 1)$ and $(-y + 3)$

[4] We set the cyclic arrangement of the factors as
$(x - 1)$, $(y - 1)$ and $(2x + 3)$, $(-y + 3)$

[5] Hence factors of the polynomial (A) are
$(x + y - 1)$ And $(2x - y + 3)$ (Retention)

(Note the argument)

Ex. 8 :- Factorize $2x^2 + 5xy + 2y^2 + x + 5y + 12$ -------- (A)

Answer :- Here D' is not equal to zero, hence polynomial is not factoriazable. We confirm this by putting $y = 0$ in (A), we get $2x^2 + x + 12$. This polynomial is not factoriazable into rational factors as $D < 0$.

Ex. 9 :- Factorize $x^2 + 2y^2 + x - y - 6$ ------------------- (A)

Answer :- Here D' is not equal to zero, hence polynomial is not factoriazable. But by above method we get the factors as $(x + 2y + 3)$ and $(x + y - 2)$.

When we check the factors we find that

$(x + 2y + 3)(x + y - 2) = x^2 + 3xy + 2y^2 + x - y - 6$

which is not the given polynomial (A). Thus for any value of 'h', $x^2 + 2hxy + x - y - 6$ will give same factors $(x + 2y + 3)$ and $(x + y - 2)$ which is ridiculous. Of course this happens only because method works irrespective of value of 'h' which is eliminated due to substitution $x = 0$ and $y = 0$.

This is drawback of the method.

However when it is guaranteed that $D' = 0$ or given that the polynomial is factoriazable the method is much simpler than the other methors.

EXERCISE

Factorize following polynomials :- (Given that $D^1 = 0$)

[1] $4x^2 - 12xy + 9y^2 + 8x - 12y + 3$.

[2] $2x^2 + 3xy - 9y^2 - 5x - 24y - 7$.

[3] $15xy + 10x + 6y + 4$.

[4] $50x^2 + 15xy - 9y^2 - 18y - 8$.

[5] $4x^2 + 4xy + y^2 - 1$.

[6] $3x^2 + 2xy - 8y^2 - 15x + 18$.

ANSWERS

[1] $(2x - 3y + 3), (2x - 3y + 1)$ [2] $(2x - 3y - 7), (x + 3y + 1)$

[3] $(5x + 2), (3y + 2)$ [4] $(10x - 3y - 4), (5x + 3y + 2)$

[5] $(2x - y + 1), (2x + y - 1)$ [6] $(x + 2y - 3), (3x - 4y - 6)$

9.7 Factorization of second degree homogeneous polynomial in three variables.

Steps :-

[1] We write the polynomial as
$ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2$ --------------- (A)

[2] Put $z = 0$ in (A), we get $ax^2 + 2hxy + cz^2$ (Elimination)
Let factors are :- $(px + qy)$ and $(kx + my)$

[3] Put $x = 0$ in (A), we get $by^2 + 2fyz + cz^2$ (Elimination)
Then factors are :- $(qy + rz)$ and $(my + nz)$

[4] Put $y = 0$ in (A), we get $ax^2 + 2gxz + cz^2$ (Elimination)
Then factors are :- $(px + rz)$ and $(kx + nz)$
(Note the argument)

[5] We set the cyclic arrangement of the factors as
$(px + qy)$, $(qy + rz)$, $(rz + px)$ and
$(kx + my)$, $(my + nz)$, $(nz + kx)$

[6] Hence factors of the polynomial (A) are
$(px + qy + rz)$ and $(kx + my + nz)$ (Retention)
(Note the argument)

Note: Polynomial (A) is factoriazable only when polynomials in step [2], [3] and [4] are factoriazable.

Ex. 10 :- Factorize $6x^2 + 8xy - 8y^2 - 11xz + 2yz + 3z^2$

Steps :-

[1] We write the polynomial as
$6x^2 + 8xy - 8y^2 - 11xz + 2yz + 3z^2$ -------------- (A)

[2] Put $z = 0$ in (A), we get $6x^2 + 8xy - 8z^2$
The factors are :- $(3x - 2y)$ and $(2x + 4y)$

[3] Put $x = 0$ (A), we get $-8y^2 + 2yz + 3z^2$
Then factors are :- $(-2y - z)$ and $(4y - 3z)$
(Note the argument)

[4] Put $y = 0$ in (A), we get $6x^2 - 11xz + 3z^2$
Then factors are :- $(3x - z)$ and $(2x - 3z)$
(Note the argument)

[5] We set the cyclic arrangement of the factors as
$(3x -2y)$, $(-2y - z)$, $(-z + 3x)$ and
$(2x + 4y)$, $(4y - 3z)$, $(-3z + 2x)$

[6] Hence factors of the polynomial (A) are

(3x - 2y - z) and (2x + 4y - 3z) (Retention)

(Note the argument)

EXERCISE

Factorize :-

[1] $6x^2 + 3xy + xz - yz - z^2$

[2] $3x^2 + 5xy + 11xz + yz - 2y^2 + 6z^2$

[3] $3oxy - 10xz - 9y^2 + 15yz - 4z^2$

[4] $24x^2 + 34xy - 7xz - 10y^2 + 19yz - 6z^2$

[5] $4xy + 14xz - 6yz - 21z^2$

[6] $-2x^2 + 11xy - 12xz + 21y^2 - yz - 10z^2$

ANSWERS

[1] (3x - z) (2x + y + z) [2] (x + 2y + 3z), (3x - y + 2z)

[3] (10x - 3y + 4z), (3y - z) [4] (3x + 5y - 2z) (8x - 2y + 3z)

[5] (2x - 3z) (2y + 7z) [6] (-x + 7y - 5z) (2x + 3y + 2z)

9.8 Factorization of second degree non homogeneous polynomial in three variables :-

Steps :-

[1] We write the polynomial as

$ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 + ux + vy + wz + t$

[2] Put y = 0, z = 0 in (A) we get $ax^2 + ux + t$ (Elimination)

Let the factors are :- (px + s), (1x + k)

[3] Put z = 0, x = 0 in (A) we get $by^2 + vy + t$

Then the factors are (qy + s), (my + k)

[4] Put x = 0, y = 0 in (A) we get $cz^2 + wz + t$,

Then the factors are (rz + s), (nz + k)

(Note the arguments in 3] and 4])

[5] We set the cyclic arrangement of the factors as :

(px + s), (qy + s), (rz + s) And (1x + k, (my + k), (nz + k)

[6] Hence the factors of the polynomial (A) are

(px + qy + rz + s) and (1x + my + nz + k)

Note :

[1] The polynomial (A) is factoriazable only when the polynomials in step. [2], [3] and [4] are factoriazable.

[2] For all values of h, g, f. The method gives same factors hence this is the drawback of the method. Thus correctness of the factors is

to ve checked by checking methods or actual product.

Ex. 11 Factories $x^2 - 3xy + 4xz + 2y^2 - 2y + 3z^2 + 5z\ x - 12$

Steps : (A)

[1] Put $y = 0, z = 0$ in (A) we get $x^2 - x - 12$

The factors are $(x - 4), (x + 3)$

[2] Put $z = 0, x = 0$ in (A) we get $2y^2 - 2y - 12,$

The factors are $(-2y - 4), (-y + 3)$

The factors are written by observing the factors in step [1].

[3] Put $z = 0\ x = 0$ in (A) we get $3z^2 + 5z - 12,$

The factors are $(3z - 4), (z + 3)$

[4] We set the cyclic arrangement of the factors as

$(x - 4), (-2y - 4), (3z - 4)$ And $(x + 3), (-y + 3), (z + 3)$

[5] Hence factors of the polynomial (A) are

$(x - 2y + 3z - 4)$ and $(x - y + z + 3)$

EXERCISE

Factorize

[1] $4x^2 + 4xy - 3y^2 + 4xz + 2yz + 8x + 8y + 4z + z^2 + 3$

[2] $12xy - 16xz + 8x - 8z^2 + 6yz + 4z$

[3] $3x^2 + 5xy - 2y^2 - 12z^2 - 5xz + 11yz + 17x - y + 14z + 10$

[4] $6x^2 + xy - 7xz - y^2 + 4z + x - 2y - 3z^2 - 1$

ANSWERS

[1] $(2x - y + z + 3)(2x + 3y + z + 1)$

[2] $(4x + 2z)(3y - 4z + 2)$

[3] $(x + 2y - 3z + 5)(3x - y + 4z + 2)$

[4] $(2x + y - 3z + 1)(3x - y + z - 1)$

9.9 Checking Method

This is the unique feature of the Vedic mathematics where factors of the polynomials can be checked easily without actual multiplication.

SUTRA : गुणितसमुच्चय: gunitasamuccayah

(The whole product will be same)

Beejank of the polynomial is sum of its coefficients. If this sum is two or more digit number then we repeatedly add the digits of sum till we get single digit as answer.

Steps :- [1] Find the Beejank of the given polynomial.

[2] Find the Beejank of each factor of the polynomial.

[3] Find Beejank of product of Bajankas in step 2]

If answers in step [1] and [3] are equal then the factors are cooect
If Beejank is zero or negative we add '9' before we compare.
Illustrations :-

[1] In Ex. 1, The beejank of given polynomial is :

$B\{x^2 + 6x + 8\} = B(1 + 6 + 8) = B(15) = 6$

The Beejank of the factors are :

$B(x + 4) = 1 + 4 = 5, B(x + 2) = 1 + 2 = 3$

The Beejank of product of these Beejanks is :

$B(5 X 3) = B(15) = 6$

As answer (6) is same the factors are corrrect.

[2] In Ex. 11 The Beejank of given polynomial is ;

$B(x^2 - 3xy + 4xz + 2y^2 - 2y + 3z^2 + 5z - x - 12 - 5yz)$

$= (1 - 3 + 4 + 2 - 2 + 3 + 5 - 1 - 12 - 5) = B(-8) = -8 + 9 = 1$

The Beejank of the factors are :

$B(x - 2y + 3z - 4) = 1 - 2 + 3 - 4 = B(-2) = -2 + 9 = 7$

$B(x - y + z + 3) = 1 - 1 + 1 + 3 = 4$

Now $B(7 X 4) = B(28) = B(10) = 1$

As answer (1) is correct the factors are correct.

Readers are suggested to check the answers of other examples.

CHAPTER 10.

CUBIC EQUATIONS

10.1 There are good methods of solving a cubic equation. But many of them are tedious as well as lengthy. VM provides the method simple and short method to find real roots. The formula used here is:

{1} पूरणापूरणाभ्याम् - puranapuranabhyam

(By completion and non completion)

{2} गुणकसमुच्चयः gunaksamuccayah

(collection of multipliers)

10.2 Ex. 1).Solve $x^3 - 6x^2 + 11x - 6 = 0.$

(1) The following is the traditional method.

Remove the second-degree term

Put $y + h$ for x

$(y + h)^3 - 6(y + h)^2 + 11(y + h) - 6 = 0.$

$y^3 + y^2(3h-6) + y(3h^2 - 12h + 11) + (h^3-6h^2+11h-6)=0.$(A)

Put coefficient of y^2 equal to zero.

$3h - 6 = 0 \qquad h = 2$

Put in (A). we get $y^3 - y = 0 \qquad y = 0$ or 1 or -1

$x = 0, 1, 2.$

(2). Method using VM.

Here we combine Paravartya with Purna formulae.

$x^3 - 6x^2 + 11x - 6 = 0.$

$x^3 - 6x^2 = -11x + 6$ ..... (B)

But. $(x - 2)^3 = x^3 - 6x^2 + 12x - 8$

$x^3 - 6x^2 = (x - 2)^3 - 12x + 8$

Put in (B). $(x - 2)^3 = x - 2$

Put $x - 2 = y$

$y^3 = y$ $y = 0$ or 1

$x = 0$ or 1 or 2.

In this met hodwe need not remove the second term.

Ex. 2) Sol ve : $x^3 + 9x^2 + 24x + 16 = 0$

$x^3 + 9x^2 = -24x - 16$ .....(A)

But $(x + 3)^3 = x^3 + 9x^2 + 27x + 27$

$x^3 + 9x^2 = -(x + 3)^3 - 27x - 27$

(A) becomes $(x + 3)^3 - 27x - 27 = -24x - 16$

$(x + 3)^3 - 3x - 11 = 0$

Put $x + 3 = y$

$y^3 - 3(y - 3) - 11 = 0$

$y^3 - 3y - 2 = 0$

$(y + 1)^2(y - 2) = 0$

$y = -1 , -1 , 2$ and $x = y - 3$

$x = -4, -4, 1$

Ex. 3). Solve $x^3 + 8x^2 + 19x + 12 = 0$

$$\text{Now, } (x+3)^3 = x^2 + 9x^2 + 27x + 27$$
$$= (x^3 + 8x^2) + x^2 + 27x + 27$$
$$= -19x - 12 + x^2 + 27x + 27$$
$$= x^2 + 8x + 15$$
$$= (x+3)(x+5)$$
$$(x+3)[(x+3)^2 - (x+5)] = 0$$
$$(x+3)(x^2 + 5x + 4) = 0$$
$$(x+3)(x+1)(x+4) = 0$$
$$x = -1, -3, -4.$$

Here formulae {1} is used.

10.3 Method of Differential Calculus :

Let, $z = px^3 + qx^2 + rx + s = 0 = (x \dotplus a)(x + b)(x + c) = A\ \ B\ \ C$

Then, $dz / dx = \Sigma AB$

$d^2z / dx^2 = (2!)\ \Sigma AB$, $(2!)$ = factorial of 2]

$d^3z / dx^3 = (3!)\ \Sigma A$ etc.

Here the formula {2} is used.

(I) Solve $z = x^3 - 4x^2 + 5x - 2 = 0$

$$dz / dx = 3x^2 - 8x + 5 = (x - 1)(3x - 5)$$

$$d^2z / dx^2 = 6x - 8 = 2(3x - 4)$$

$$= 2[(x-1)+(x-1)+(x-2)]$$

$$z = (x-1)(x-1)(x-2) = 0$$

$$x = 1, 1, 2.$$

Solve : $x^3 + 6x^2 + 9x + 4 = 0.$ ———————— (A)

Now, $(x+2)^3 = x^3 + 6x^2 + 12x + 8$

$$x^3 + 6x^2 = (x+2)^3 - 12x - 8$$

Put in (A)

$$(x+2)^3 - 12x - 8 + 9x + 4 = 0$$

$$(x+2)^3 - 3x - 4 = 0$$

Put $x + 2 = y$

$$y^3 - 3(y-2) - 4 = 0$$

$$y^3 - 3y + 2 = 0$$

$$y^3 - y^2 + y^2 - y - 2y + 2 = 0$$

$$y^2(y-1) + y(y-1) - 2(y-1) = 0$$

$$(y-1)(y^2 + y - 2) = 0$$

$$(y-1)(y+2)(y-1) = 0$$

$$y = 1, 1, -2$$

$$x = -1, -1, -4$$

10.4 Biquadrati cEquati ons

Here also, formula { 1 } i sused.Ex. 1).

Solve $x^4 + 4x^3 + 25x^2 - 16x + 84 = 0$ —— (A)

Now $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

$= (25x^2 + 16x - 84) + (6x^2+4x+1)$ from (A)

$= 31x^2 + 20x - 8x$

$= (x + 1)(31x - 11) - 72$

Put $x + 1 = y$

$y^4 - 31y^2 + 42y + 72 = 0$

$y = -1 , 3 , 4$ or -6

$x = -2 , 2 , 3$, or -7

Ex. 2). Solve : $(x+7)^4 + (x+5)^4 = 706$

Here we use लोपनस्थापनाभ्याम् (Lopansthapanabhyam)

By alternate eli minationand retention rule of V.M

Average of $x + 7$ & $x + 5$ is $x + 6$.

Let, $x + 6 = y$

$(y+1)^4 + (y-1)^4 = 706$

$y^4 + 6y^2 - 352 = 0$ $y^2 = 16$ or -22

$y = 4$ or (-22)

The second root is *complex*. Hence it is omitted.

$x = -2$ or -10.

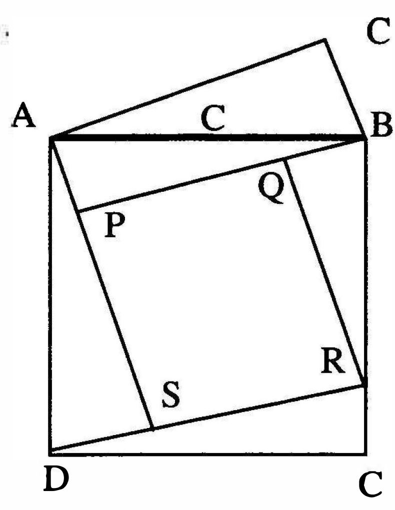
❖ ❖ ❖

Part - II Geometry
CHAPTER 11.
Pythagoras' & Apollonius' Theorems.

11.1 Four different proofs of Pythagoras theorem are given below.

Statement :- In a right angled triangle the square of hypotenuse is equal to the sum of the squares of remaining two sides.

Proof (1).



Four triangles congruent with $\triangle ABC$ are arranged in such a way that $\square ABCD$ and PQRS are squares.

Here $AB = c$, $AC = b$, $CB = a$

$$PQ = QR = RS = SP = b - a$$

Here we have

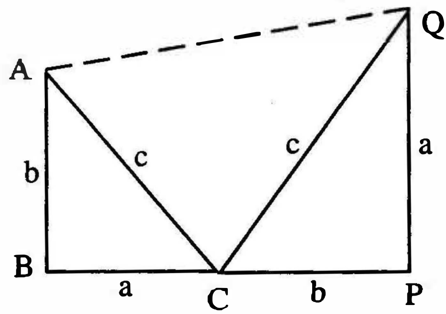
A (ABCD) = Area of 4 triangles + A (PQRS)

$$c^2 = 4 \times 1/2\,ab + (b - a)^2$$

$$c^2 = 2ab + b^2 - 2ab + a^2$$

$$c^2 = a^2 + b^2$$

Proof : (2)



In the adjoining Fig. ABC $\cong$ CPQ

These two similar triangles are arranged such that B - C - P and ACQ = 90^0

and □ABPQ is trapezium.

Here we have

$$A(ABPQ) = \text{Sum of areas of 3 triangles.}$$

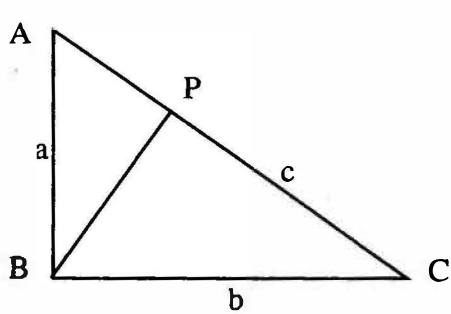
$$1/2\,(a + b)(a + b) = 1/2\,ab + 1/2\,ab + 1/2\,c^2$$

$$1/2\,(a^2 + 2ba + b^2) = 1/2\,ab + 1/2\,ab + 1/2\,c^2$$

$$a^2 + b^2 = c^2.$$

Q.E.D

Proof : (3)



In the adjoining Fig. :

Three right angled triangles are shown. These are similar. We know areas of similar triangles are proportional to square of corresponding sides.

$$\frac{A(\triangle BPA)}{A(\triangle ABC)} = \frac{AB^2}{AC^2} \qquad (1)$$

and

$$\frac{A(\triangle CPB)}{A(\triangle ABC)} = \frac{BC^2}{AC^2} \qquad (2)$$

Add (1) and (2).

$$\frac{A(\triangle BPA) + A(\triangle CPB)}{A(\triangle ABC)} = \frac{AB^2 + BC^2}{AC^2}$$

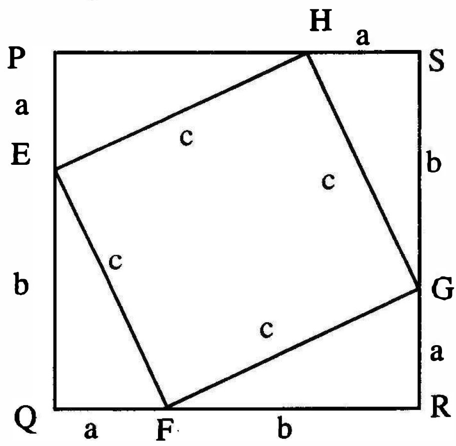
i.e.

$$\frac{A(\triangle ABC)}{A(\triangle ABC)} = \frac{AB^2 + BC^2}{AC^2}$$

$$\therefore \quad \frac{AB^2 + BC^2}{AC^2} = 1$$

i.e. $a^2 + b^2 = c^2$. Q.E.D

4). Proof :



In the adjoining figure.

Four triangles congruent with $\triangle ABC$ are arranged such that $\square PQRS$

and EFGH are squares. By area addition

We have,

$$A(\square PQRS) = 4\,A(\triangle ABC) + A(\square EFGH)$$

$$(a+b)^2 = 4 \times 1/2\, ab + c^2$$

$$\therefore \quad a^2 + 2ab + b^2 = 2ab + c^2$$

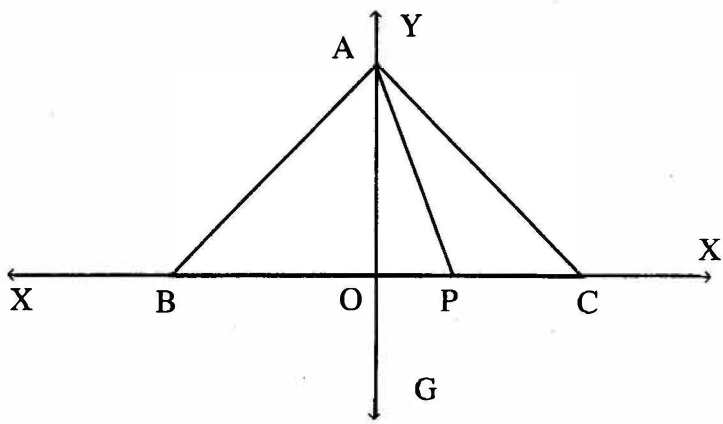
$$\therefore \quad a^2 + b^2 = c^2.$$

20.2 APOLLONIUS' THEOREM

Statement :- In any $\triangle ABC$,

$AB^2 + AC^2 = 2(AP^2 + PC^2)$

where AP is median of $\triangle ABC$.



Proof :- Consider any $\triangle ABC$, where AP is median

We construct coordinate axis x o x and through base BC and
y o y axis through point A

Let $OA = p$, $OB = m$, $OP = n$ then $BP = m + n$

Thus $OC = OP + PC = n + m + n = 2n + m$

Hence coordinates of the vertices of $\triangle ABC$ and point p are

A $(0, p)$, B $(-m, 0)$, C $(2n + m, 0)$ and p $(n, 0)$.

Now by distance formula

$AB^2 = m^2 + p^2$,

$AC^2 = (2n+m)^2 + p^2 = 4n^2 + 4mn + m^2 + p^2$

And $\quad AP^2 = n^2 + p^2$,

$PC^2 = (n + m)^2 = n^2 + 2mn + m^2$

Now $\quad AB^2 + AC^2 = 2m^2 + 2p^2 + 4n^2 + 4mn$

And $\quad AP^2 + PC^2 = m^2 + p^2 + 2n^2 + 2mn$

Thus $\quad AB^2 + AC^2 = 2(AP^2 + PC^2)$

Q.E.D.

CHAPTER 12.

TRIPLETS

12.1 Definition :- A set of three real numbers x , y and z satisfying the equation $x^2 + y^2 = z^2$ is called TRIBHUJANK or a Triplet. It is written as [x, y, z]

[3, 4,5] is a triplet as $3^2 + 4^2 = 5^2$

[5,12,13] is a triplet.

[1,2,3] is not a triplet.

The order in which x, y, appear in [x , y , z] is worth discussion. Hence we have

(1) [x , y , z] = [y , x , z] also,

(2) [x , y , z] [y , z , x] [z , x , y]

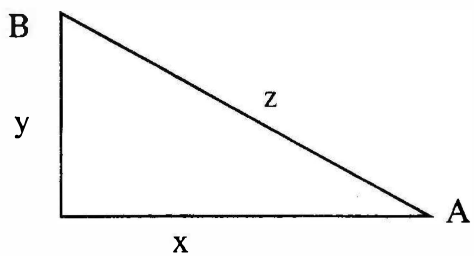
(3) [x , y , z] = [- x , y , z] = [x , - y , z] etc.

(4) [x , y , z] exists if and only if $x^2 + y^2 - z^2 = 0$

(5) [x , y , z] = [a , b , c] x = a , y = b , z = c

Hereafter, we shall not be free to use any sequence of x , y and z for the reason that triplet will be associated with an angle.

12.2 It is clear that when $x^2 + y^2 = z^2$, where x , y , z are the sides of a plane right angled triangle. Assume that angle between x and z is A and that between y and z is B, z being the hypotenuse.



Definition :- Every triplet [x , y , z] is associated with angle A opposite to side y. and is put as,

T (A) = [x , y , z]

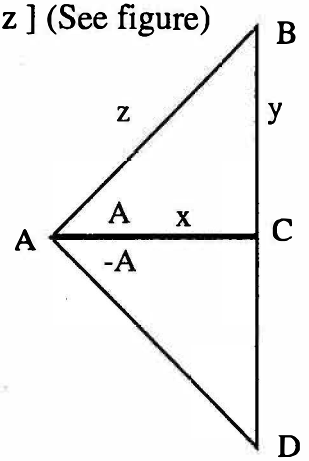
obviously, T (B) = [y , x , z] where and B = 90^0 - A

(Note that we can not write T (90^0) = [x , z , y] for the simple reason that, ($x^2 + z^2 \ngeq y^2$)

Thus, when T (A) = [x , y , z]

then T (90^0 - A) = [y , x , z].

12.3 Triplet for angle (- A) : If T (A) = [x , y , z] then T (-A) = [x ,-y, z] (See figure)



Here T (D) = [-y , x , z] , (see figure).

Note:

[1] T (A) + T (B) T (A+B) for any two angles A and B

[2] T is just a notation for angle A. T (A) represents a triplet associated with A. T (A) is neither a function nor an operator.

[3] Triplet [x , y , z] is not unique for angle A. (147)

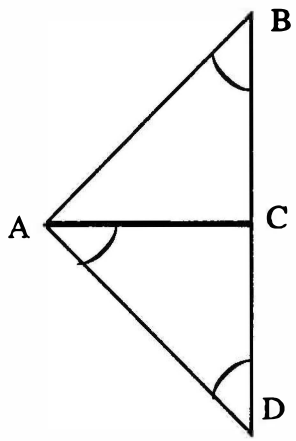
For, T (A) = [x , y , z] = [k x , k y , k z] for any positive real number k.

[4] Angle A is not unique for a triplet [x , y , z]. So T (A) = [x , y , z] is not unique on either side. The context, wherever necessary, will reveal the exact triplet.

Ex :- Let z = 5 and x = 4 then, $y^2 = (z+x)(z-x)=(5+4)(5-4)=9$

This gives y = 3 so satisfying $x^2 + y^2 = z^2$.

We get two values of y thereby generating two triangles.

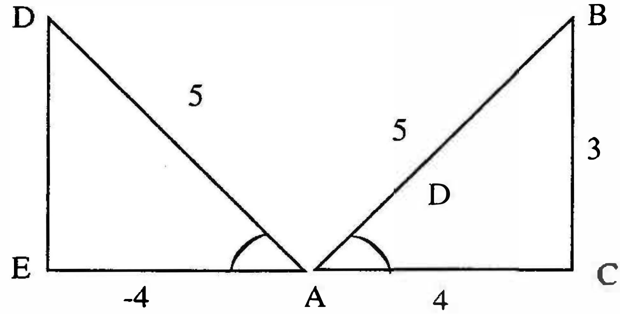


Here T (A) = [4,3,5], T (B) = [3,4,5] , T (D) = [-3,4,5]

Numerically, angles B and D are equal, but their triplets vary.

Similarly if y = 3 and z = 5 then x = 4

Also, CAB has triplet [4,3,5] but EAD has triplet [-4,3,5] though these angles have same measures.



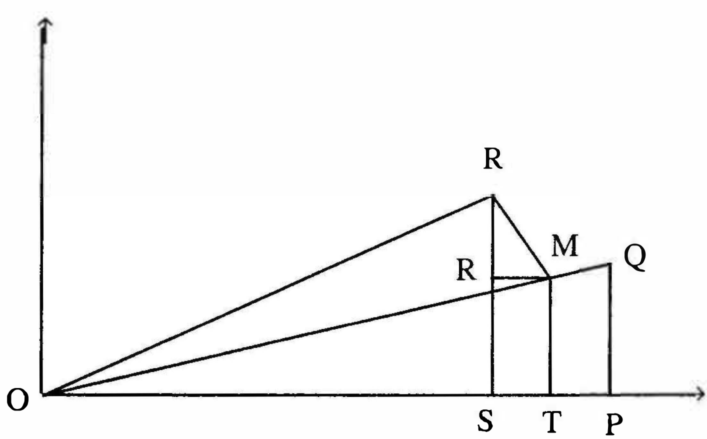
12.4 :- Triplets For Sum of Two Angles.

Theorem :- If $T(A) = [x_1, y_1, z_1]$, $T(B) = [x_2, y_2, z_2]$ then,

$$T(A+B) = [x_1x_2 - y_1y_2, x_2y_1 + y_2x_1, z_1z_2]$$

Assume that A and B are positive angles.

Proof :- Construct two right angled triangles OPQ and OMR with angles A and B as shown in figure. Draw RS and MT perpendicular on OP and MN perpendicular on RS.



Here in triangle OPQ, we have POQ=A and $T(A)=[x_1, y_1, z_1]$

Hence $OP = x_1$, $PQ = y_1$, $OQ = z_1$.

Similarly, in triangle OMR, we have QOR = B and

$T(B) = [x_2, y_2, z_2]$

Hence $OM = x_2$, $RM = y_2$, $OR = z_2$.

Now from figure, SOR = A + B, hence

$T(A+B) = [OS, SR, OR]$

We find these lengths in terms of triplets of angle A and B.

Now from figure, OPQ is similar to OTM

Hence 1) $(OT / OP) = (OM / OQ)$ Thus $OT = (x_1 x_2 / z_1)$

and 2) $(TM / PQ) = (OM / OQ)$ Thus $TM = (y_1 x_2 / z_1)$

Now NRM = SOR = A, hence OPQ is similar to RNM

Hence 3) $(NM / PQ) = (RM / OQ)$

Thus $NM = (y_1 y_2 / z_1)$ and

4) $(RN / OP) = (RM / OQ)$

Thus $RN = (x_1 y_2 / z_1)$

Again from figure,

$OS = OT - ST = OT - NM = (x_1 x_2 - y_1 y_2) / z_1$ and

$SR = SN + NR = TM + NR = (y_1 x_2 + x_1 y_2) / z_1$

Thus $T(A+B) = [x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2, z_1 z_2]$

12.5 Triplets for the difference of angles :

Theorem : If $T(A) = [x_1, y_1, z_1]$, $T(B) = [x_2, y_2, z_2]$

Then $T[A - B] = [x_1 x_2 + y_1 y_2, y_1 x_2 - y_2 x_1, z_1 z_2]$

Proof :- Construct two right angled triangles OSR & OQM

Now $\angle NQM = \angle SOQ = A$. hence $\triangle QNM$ is similar to $\triangle OSR$.

Hence 1) (NM/SR)=(QM/OR) hence NM=(y_1y_2 / z_1)

2) (QN / OS) = (QM / OR) hence QN = (x_1y_2 / z_1)

Again from figure, OT=OP+PT = OP + NM = ($x_1x_2 + y_1y_2$) / z_1

TM = PN = PQ - QN = ($y_1x_2 - y_2x_1$) / z_1

Thus T (A - B) = [$x_1x_2 + y_1y_2$, $y_1x_2 - y_2x_1$, z_1z_2]

Examples :

(1) Let T (A) = [3 , 4 , 5] and T (B) = [12 , 5 , 13]

Then T (A + B) = [312 - 45 , 412 - 35 , 513] = [16,4, 65]

T (A-B) = [312 + 45, 412 + 35 , 513]=[56, 33, 65]

But lengths of hypotenuse of A-B and B are equal & each should 13. Hence divide by 5,

$\therefore$T (A-B) = [56/5 , 33/5 , 13].

(2)Find the triplets for A+B and A-B, given that,

T(A)=[1,2,5] and T (B) = [3 , 2 , 13].

(This work is left to the reader.)

12.6 We use the above two results to find the triplets for angle 2A and angle zero

We have T (A $\pm$ B) = [$x_1x_2 \mp y_1y_2$, $y_1x_2 \pm x_1y_2$, z_1z_2]

(1) Put B = A in A+B then [$x_2 y_2 z_2$] [$x_1 y_1 z_1$]

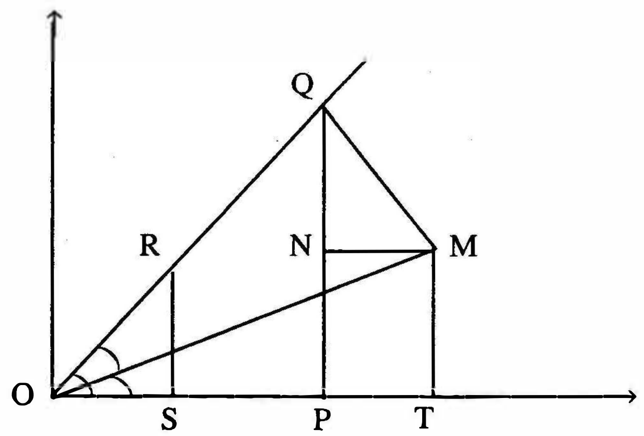
$\therefore$ T (2A) = [$x_1^2 - y_1^2$, $2x_1y_1$, z_1^2]

(2) Put B = A in triplet of A - B

(150) with $\angle SOR = A$ and $\angle MOQ = B$.

Draw MT on OP, MN on PQ and extend it to meet OQ at R.

Draw RS on OP as shown in the figure.



Here in $\triangle OSR$ we have $\angle SOR = A$ and $T(A) = [x_1, y_1, z_1]$

Hence $OS = x_1$, $RS = y_1$, $OR = z_1$. Si milarlyin $\triangle OQM$ we have $\angle MOQ = B$ and $T(B) = [x_2, y_2, z_2]$. Hence $OQ = x_2$, $QM = y_2$, $OM = z_2$.

Now from figure $\angle TOM = A - B$.

Hence in $\triangle OTM$ $T(A - B) = [OT, TM, OM]$

We find these lengths in terms of triplets of A and B.

Now, $\triangle OPQ$ is similar to $\triangle OSR$.

Hence 1) $(OP / OS) = (OQ / OR)$, hence $OP = (x_1 x_2 / z_1)$

2) $(PQ / SR) = (OQ / OR)$, hence $PQ = (y_1 x_2 / z_1)$

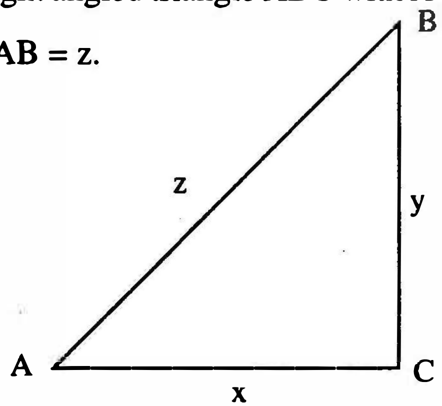
$$\therefore T(0) = [x_1^2 + y_1^2, 0, z_1^2]$$
$$= [z_1^2, 0, z_1^2]$$
$$= [1, 0, 1]$$

One may feel how T (0) exists when there is no triangle with an angle of measure zero and hence, T (0) = [1 , 0 , 1] seems to be self-contradictory. But this is not the case. Firstly we have defined a triplet [x , y , z] which satisfies the condition $x^2 + y^2 = z^2$.[1 , 0 , 1] satisfies this condition. Secondly measures 0^0, 180^0 , 270^0 , 360^0 are the angles of triangle in their limiting cases.

Hence we have T (0) = [1 , 0 , 1] etc.

12.7. Triplets for 90^0, 180^0, 270^0 and 360^0.

Consider a right angled triangle ABC with AC = x , CB = y and hypotenuse AB = z.



then

$$T(A) = [x, y, z], T(B) = [y, x, z]$$
$$T(A+B) = [0, x^2+y^2, z^2]$$

$= [\,0\,,\, z^2\,,\, z^2\,]$

$= [\,0\,,\,1\,,\,1\,]$

But A+B = 90^0 $\therefore$ T (90^0) = [0 , 1 , 1]

Using formula triplet of angle 2A we get

T (180^0) = [-1 , 0 , 1] and then T (360^0) = [1 , 0 , 1].

Note that T (0^0) = [1 , 0 , 1] = T (360^0)

Also since T (180^0) = [-1 , 0 , 1] and T (90^0) = [0 , 1 , 1]

T (270^0) = T ($180^0 + 90^0$) = [0 , -1 , 1].

12.8. Triplet for angle (A / 2).

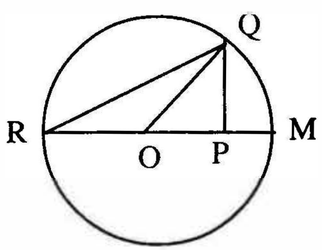
Let A = [x , y , z] so that $z^2 = x^2 + y^2$.

Let OP = x , PQ = y , OQ = z

Draw a circle with center O and radius OQ = z. $\angle$A and $\angle$PRQ are on the same chord QM and $\angle$POQ = A,

$\therefore$ $\angle$PRQ = A/2

Also, OR = OQ = z



Therefore the triplet of A/2 = [RP , PQ , RQ]

That is, T (A/2) = [x + z , y , $\sqrt{(x+z)^2 + y^2}$]

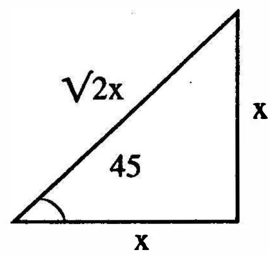
This result is used to find triplet for 45^0.

$T(90^0) = [0, 1, 1]$

$T(45^0) = [0, +1, 1, \sqrt{(0+1)^2 + 1^2}] = [1, 1, \sqrt{2}]$.

Another proof :

$$T(45^0) = [x, x, \sqrt{2}\,x]$$
$$= [1, 1, \sqrt{2}]$$

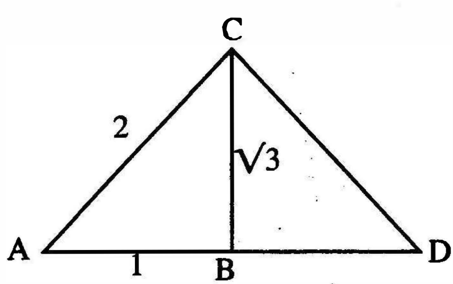


21.9. Triplets for 60^0 and 30^0.

Consider an equilateral triangle with each side of measure 2.

Now AC = 2 and AB = 1 hence CB = 3

For A = 60^0, $T(60^0) = [AD, DC, AC] = [1, \sqrt{3}, 2]$



Similarly for ACD = 30^0 ,

$T(30^0) = [1+2, 3, \sqrt{(1+2)^2 + (\sqrt{3})^2}] = [3, \sqrt{3}, 2\sqrt{3}]$

$= [3, 1, 2]$, after dividing by 3 .

12.10 Now, we are in a position to find the triplets for 120^0, 135^0, 150^0 and also for the rarely used angles 15^0, 75^0. We state the results only. The verification is left to the reader.

$$T(75^0) = [\sqrt{3} - 1, \sqrt{3} + 1, 2\sqrt{2}]$$

$$T(120^0) = [-1, \sqrt{3}, 2]$$

$$T(135^0) = [-1, 1, \sqrt{2}]$$

$$T(150^0) = [-\sqrt{3}, 1, 2]$$

$$T(300^0) = [1, -\sqrt{3}, 2]$$

$$T(15^0) = [\sqrt{3} + 1, \sqrt{3} - 1, 2\sqrt{2}]$$

$$T(105^0) = [1 - 3, 1 + 3, 2\sqrt{2}]$$

12.11 Summary

If $\quad T(A) = [x_1, y_1, z_1]$ & $T(B) = [x_2, y_2, z_2]$

then for positive angles A and B,

$$T(A + B) = [x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2, z_1 z_2]$$

$$T(A - B) = [x_1 x_2 + y_1 y_2, y_1 x_2 - x_1 y_2, z_1 z_2]$$

$T(0^0) = [1, 0, 1]$	$T(30^0) = [\sqrt{3}, 1, 2]$
$T(90^0) = [0, 1, 1]$	$T(45^0) = [1, 1, \sqrt{2}]$
$T(180^0) = [-1, 0, 1]$	$T(60^0) = [1, \sqrt{3}, 2]$
$T(270^0) = [0, -1, 1]$	$T(360^0) = [1, 0, 1]$

$$T(2A) = [\, x_1^2 - y_1^2 ,\ 2x_1 y_1 ,\ z_1^2 \,]$$

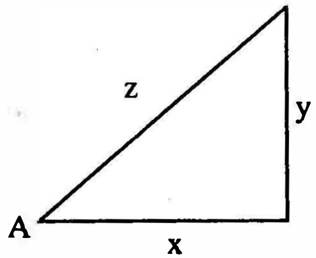
$$T(A/2) = [\, x + z ,\ y ,\ \sqrt{[(x+z)^2 + y^2} \,]$$

❖ ❖ ❖

CHAPTER 13.

TRIGONOMETRICAL RATIOS

13.1 Here we try to discuss how Trigonometry, one of the basic branches of mathematics, can be developed using Vedic Mathematics. It will be evident to the reader later on that Vedic approach to trigonometry is more easy and lucid than any other approach. The concept of triplets (TRIBHUJANK) is a base of entire Trigonometry.



Definition:If T (A) =[x,y,z]for any angle A in first quadrant then

[1] $\sin A = y / z$ [2] $\cos A = x / z$ [3] $\tan A = y / x$

[4] $\operatorname{cosec} A = z / y$ [5] $\sec A = z / x$ [6] $\cot A = x / y$.

Note :

1. When A lies in second quadrant, T (A) = [- x , y , z]

2. When A lies in third quadrant, T (A) = [- x , - y , z]

3 When A lies in forth quadrant, T (A) = [x , - y , z]

4. The definition of trigonometric ratio will vary accordingly.

We note that sine, cosine, tangent etc. are all circular functions. If A is the measure of an angle then their is one and only one value of sin A. This gives rise to a function from the set of measures of angles which are real numbers to a set of real numbers in form of a ratio y/z. This function is called sine function. Its domain is R , i.e. set of real numbers. Similarly for the other functions.

13.2. Relations between trigonometric Ratios

From definitions above we can state following relations:

[1] (sin A / cos A) = tan A [2] (cos A / sin A) = cot A

[3] (1 / tan A) = cot A [4] (1 / sin A) = cosec A

[5] (1 / cos A) = sec A

13.3. Now we shall establish some basic results .

To prove : [1] $\sin^2 A + \cos^2 A = 1$ [2] $1 + \tan^2 A = \sec^2 A$

[3] $1 + \cot^2 A = \text{cosec}^2 A$

Proof : Let, T (A) = [x , y , z] then $\sin A = y / z$, $\cos A = x / z$
hence $\sin^2 A + \cos^2 A = (y^2 / z^2) + (x^2 / z^2)$

$$= (y^2 + x^2) / z^2 = (z^2 / z^2) = 1$$

Similarly,

$1+\tan^2 A=1+(y^2/x^2)=(x^2+y^2)/x^2=z^2/x^2=(z/x)^2 = \sec^2 A$

$1 + \cot^2 A = 1 + (x^2 / y^2)= (y^2 + x^2) / y^2 = z^2 / y^2 = (z / y)^2 =$ $\text{cosec}^2 A$

Illustrative Examples :

Ex.[1].

If sin A = 11 / 61, obtain tan A and cos A.

Answer : Let,T (A) = [x , 11 , 61]

$x^2 + 121 = 3721$ $x^2 = 3600$ and $x = 60$.

T (A) = [60 , 11 , 61] and cos A = 60 / 61, tan A = 11 / 61.

Ex.[2]. sin = 3 / 2 , lies in 2nd quadrant. Obtain cos and tan .

Let ,T () = [x , 3 , 2]

$x^2 = 4 - 3 = 1$

x = 1. Since is in II quadrant, x is negative.

T () = [-1 , 3 , 2]

andcos = -1 / 2, tan = - 3.

Ex.[3]. If $\tan\theta = 1/\sqrt{7}$, find the value of

$(\text{cosec}^2\theta - \sec^2\theta) / (\text{cosec}^2\theta + \sec^2\theta)$

Let T () = [$\sqrt{7}$, 1 , z] then $z^2 = 7 + 1 = 8$ and $z = \sqrt{8}$

T θ = [$\sqrt{7}$, 1 , 8] thus cosecθ = $\sqrt{8} / 1$, sec = $\sqrt{8} / 7$

$(\text{cosec}^2\theta - \sec^2\theta)/(\text{cosec}^2\theta + \sec^2\theta) =$

(8-8/7)/(8+8/7) = 48 / 64 = 3 / 4.

13.4 Trigonometric ratios for standard angles using triplets

(1) T (0^0) = [1 , 0 , 1], sin 0^0 = 0 / 1 = 0 and cos 0^0 = 1 / 1 = 1.

(2) T (30^0) = [3 , 1 , 2], sin 30 = 1 / 2 and cos 30 = 3 / 2.

(3) T (45^0) = [1 , 1 , 2], sin 45 = 1 / 2 and cos 45 = 1 / 2.

(4) T (60^0) = [1 , 3 , 2], sin 60 = 3 / 2 and cos 60 = 1 / 2.

(5) T (90^0) = [0 , 1 , 1], sin 90 = 1/ 1 = 1 and cos 90 = 0 / 1 = 0

(6) T (180^0) = [-1 , 0 , 1], sin 180 = 0 and cos 180 = -1

The other trigonometric ratios can be obtained similarly. We summarize these results, (all angles below are in degrees)

A	0	30	45	60	90	180	270	360
sin A	0	$1\sqrt{2}$	$1\sqrt{2}$	$3\sqrt{2}$	1	0	-1	0
cos A	1	$3\sqrt{2}$	$1\sqrt{2}$	$1\sqrt{2}$	0	-1	0	1

13.5 Trigonometric Ratios for supplementary and complementary angles

Angles A and B are supplementary if $A + B = 90^0$

and complementary if $A + B = 180^0$

Note

If $T(A) = [x_1, y_1, z_1]$ and $T(B) = [x_2, y_2, z_2]$ then

(a) $T(A + B) = [x_1x_2 - y_1y_2, y_1x_2 + x_1y_2, z_1z_2]$ and

(b) $T(A - B) = [x_1x_2 + y_1y_2, y_1x_2 - x_1y_2, z_1z_2]$

(c) $T(2A) = [x_1^2 - y_1^2, 2x_1 y_1, z_1^2]$

(d) $T(A/2) = [x_1 + z_1, y_1, (x_1 + z_1)^2 + y_1^2]$

(1) To prove: $\sin(90^0 - A) = \cos A$, $\cos(90^0 - A) = \sin A$ and

$\tan(90^0 - A) = \cot A$

Proof : Let $T(A) = [x, y, z]$, $T(90^0) = [0, 1, 1]$

then $T(90^0 - A) = [y, x, z]$ { Refer Note }

Now $\sin(90^0 - A) = x/z$, and $\cos A = x/z$

$\sin(90^0 - A) = \cos A$

Also $\cos(90^0 - A) = y/z = \sin A$ and $\tan(90^0 - A) = x/y = \cot A$

(2) To prove: $\cos(90^0 + A) = -\sin A$ and $\sin(90^0 + A) = \cos A$

Proof : We have $T(90^0) = [0, 1, 1]$, $T(A) = [x, y, z]$

$T(90^0 + A) = [-y, x, z]$ {Refer Note}

$\cos(90^0 + A) = -y/z = -\sin A$

$\sin(90^0 + A) = x/z = \cos A$

(3) To prove : $\sin(180^0 - A) = \sin A$, $\sin(180^0 + A) = -\sin A$

and $\sin(360^0 - A) = \sin A$

Proofs left for the reader.

(4) To prove : sin (- A) = - sin A

cos (- A) = cos A

Proof : we have, T (0^0) = [1 , 0 , 1]

Let , T (A) = [x , y , z]

T (0 - A) = [x , - y , z]

i.e. T (- A) = [x , - y , z]

sin (- A) = - y/z = - (y/z) = - sin A and

cos (- A) = x/z = cos A.

13.6 Illustrative Examples :

(1) Show that sin 120^0 = 3 / 2 and cos 120^0 = -1 / 2.

120^0 = 90^0 + 30^0 hence T (120^0) = T (90+30)

Now T (90^0) = [1, 0 , 1] and T (30^0) = [1, 3 , 2]

hence T (120^0) = [-1, 3 , 2] and we get

sin (120^0) = 3 / 2 and cos (120^0) = -1 / 2.

(2) If sin A=12 / 13 , sin B=4/5 find sin (A - B), sin (A + B) , tan (A + B)

Here T (A) = [5 , 12 , 13] , T (B) = [3 , 4 , 5]

T (A+B) = [-33 , 56 , 65] , T (A-B) = [63 , 16 , 65]

sin (A+B) = 56 / 65 , tan (A+B) = -56 / 33 , sin (A-B) = 16 / 65.

(3) If tan A = 12 / 5 find sin 2A and sin (A / 2)

Here T (A) = [5,12,13] , hence T (2A) = [-119,120,169]

sin 2A = 120 / 169

Also T (A / 2) = [17 , 12 , $\sqrt{433}$]

sin (A / 2) = 12 / $\sqrt{433}$

(4) If tan A = 5/6 and tan B=1 / 11 show that tan (A + B) = 1

Here T (A) = [6 , 5 , $\sqrt{61}$] , T (B) = [11 , 1 , $\sqrt{122}$]

T (A+B) = [61 , 61 , 612] = [1 , 1 , $\sqrt{2}$]

tan (A + B) = 1 / 1 = 1. It is obvious that A + B = π/ 4.

(5) Show: cos (2 π/ 3) tan (3 π/4)-2cos (5 π/ 6) sin (2 π/ 3) = 2

T (2 π/ 3) = T (π- π/ 3) = [-1 , $\sqrt{3}$, 2] ,

T (3 π/ 4) = T (π - π/ 4) = [-1 , 1 , $\sqrt{2}$] ,

T (5 π/ 6) = T (π - π/ 6) = [-$\sqrt{3}$, 1 , 2]

LHS = (-1 / 2)(-1) + ($\sqrt{3}$)($\sqrt{3}$ / 2) = 2 = RHS.

(6) Prove : sec (π/4 +) . sec (π/4 -) = 2 sec 2θ

Proof : T (π/4) = [1 , 1 , $\sqrt{2}$]

Let, T (θ) = [x , y , 1]

T (π/4 + θ) = [x - y , x + y , $\sqrt{2}$]

T (π/4 - θ) = [x + y , x - y , $\sqrt{2}$]

T (2θ) = [x^2 - y^2 , 2 x y , 1]

LHS = ($\sqrt{2}$ / x-y) ($\sqrt{2}$/x+y)=2/(x^2 -y^2)=2 sec2θ = RHS.

(7) Prove : sin 105^0 + cos 105^0 = cos 45^0

Proof :T (90^0) = [0 , 1 , 1], T (15^0) = [3 + 1 , 3 - 1 , 22]

hence T (105^0) = T (90^0 + 15^0) = [1 - 3 , 1 + 3 , 22]

LHS = (1 + 3) / (22) + (1 - 3) / (22)

= 2 / (22)

= 1 / 2, as T (45) = [1 , 1 , 2]

= cos 45^0 = RHS.

(8) Show that sin 420^0 cos 390^0 + cos (-300^0) sin (-330^0) = 1

Hint : T (420^0) = T (360^0 + 60^0) = [1 , $\sqrt{3}$, 2]

$T(390^0) = T(360^0 + 30^0) = [\sqrt{3}, 1, 2]$

$T(300^0) = T(360^0 - 60^0) = [1, -\sqrt{3}, 2]$

$T(330^0) = T(360^0 - 30^0) = [\sqrt{3}, -1, 2]$

Also $\cos(-300^0) = \cos(300^0)$ and $\sin(-330^0) = -\sin(330^0)$

LHS = $(\sqrt{3}/2)(\sqrt{3}/2) + (1/2)(1/2) = (3/4) + (1/4) = 1$ = RHS.

(9) Show that $\sin 1460^0 + \cos 1190^0 = 0$

Since $1460^0 = (4 \times 360^0) + 20$

$T(1460^0) = T(20^0) = [x, y, z]$, say

Also $T(1190^0) = T(3 \times 360^0 + 90 + 20)$

$= T(90^0 + 20^0) = [-y, x, z]$

LHS $= y/z + (-y/z) = 0$ = RHS.

(10) If $\cos A = 11/61$, $\sin B = 4/5$ and A,B are both acute angles, find

$\sin^2(A-B)/2$ and $\cos^2(A+B)/2$.

Here $T(A) = [11, 60, 61]$, $T(B) = [3, 4, 5]$

$T(A+B) = [-207, 224, 305]$

$T(A-B) = [273, 136, 305]$

$T[(A+B)/2) = [98, 224, \sqrt{2 \times 305 \times 98}]$

$T[(A-B)/2) = [578, 136, \sqrt{2 \times 305 \times 578}]$

$\sin^2[(A-B)/2] = [136/(\sqrt{2 \times 305 \times 578})]^2 = 16/305$

$\cos^2[(A+B)/2] = [98/(\sqrt{2 \times 305 \times 98})]^2 = 49/305$

(11) Prove $(1 - \tan A)/(1 + \tan A) = (\cot A - 1)/(\cot A + 1)$

Let, $T(A) = [x, y, z]$

$\tan A = y/x$, $\cot A = x/y$

LHS $= [1-(y/x)] / [1+(y/x)] = x-(y/x)+y = (x/y)-1/(x/y)+1$

$= (\cot A - 1) / (\cot A + 1) = \text{RHS}.$

(12) Show : $\sin 2A / (1 + \cos 2A) = \tan A$

Let, $T(A) = [x, y, 1]$

$T(2A) = [x^2 - y^2, 2xy, 1]$

$\text{LHS} = 2xy / [1 + (x^2 - y^2)]$ But $x^2+y^2 = 1$

$= 2xy / 2x^2 = y/x = \tan A = \text{RHS}.$

(13) Show : $(\sec A - \tan A)^2 = (1 - \sin A) / (1 + \sin A)$

Let, $T(A) = [x, y, 1]$

$\text{LHS} = (1/x - y/x)^2 = (1 - y)^2 / x^2 = (1 - y)^2 / (1 - y^2)$

$= (1-y) / (1+y) = (1-\sin A) / (1 + \sin A) = \text{RHS}$

(14) Show $(\tan A + \tan B) / (\cot A + \cot B) = \tan A \tan B$

Proof : Let, $T(A) : (x_1, y_1, 1)$, $T(B) : (x_2, y_2, 1)$

$\text{LHS} = [(y_1 / x_1) + (y_2 / x_2)] / [(x_1 / y_1) + (x_2 / y_2)]$

$= [(y_1x_2+x_1y_2)/ (x_1x_2)] / [(x_1y_2+y_1x_2) / (y_1y_2)]$

$= (y_1y_2) / (x_1x_2) = (y_1/ x_1) (y_2/ x_2)$

$= \tan A \tan B = \text{RHS}.$

(15) Prove : $\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B$

Let $T(A) = [x_1, y_1, 1]$, $T(B) = [x_2, y_2, 1]$

$T(A+B) = [x_1x_2 - y_1y_2, x_2y_1 + y_2x_1, 1]$

$T(A-B) = [x_1x_2 + y_1y_2, x_2y_1 - y_2x_1, 1]$

$\text{LHS} = (y_1x_2 + x_1y_2)(y_1x_2 - x_1y_2)$

$= y_1^2x_2^2 - x_1^2y_2^2 = y_1^2(1 - y_2^2) - (1 - y_1^2)y_2^2$

$= y_1^2 - y_2^2$

$= \sin^2 A - \sin^2 B = \text{RHS}.$

(16) Show : $[1 + \tan^2(45^0 - A)] / [1 - \tan^2(45^0 - A)] = \operatorname{cosec} 2A$

We have $T(45^0) = [1, 1, \sqrt{2}]$, Let $T(A) = [x, y, 1]$

$T(45^0 - A) = [x+y, x-y, 1]$ and $T(2A) = [x^2 - y^2, 2xy, 1]$

$LHS = \{1+[(x-y)/(x+y)]^2\}/\{1-[(x-y)/(x+y)]^2\}$

$= 1/2xy = \text{cosec } 2A = RHS$

(use $x^2 + y^2 = 1$)

(17) Solve the equation : $3\cot^2 A + 5 = 7\text{cosec } A$

Let, $T(A) = [x, y, 1]$ then $3\cot^2 A + 5 = 7\text{cosec } A$ gives

$3(x/y)^2 + 5 = 7(1/y)$ But $x^2 + y^2 = 1$ hence

$3(1-y^2)/y^2 + 5 = 7/y$ $2y^2 - 7y + 3 = 0$

Solving we get $y = 1/2$ or 3. $x = \sqrt{1-y^2} = \sqrt{3}/2$ or $x = \sqrt{-8}$

$x = \sqrt{-8}$ is rejected as it is imaginary

$T(A) = (\sqrt{3}/2, 1/2, 1)$ $\sin A = 1/2$, $A = 30^0$

13.6 Trigonometric Values For 18^0 and 36^0.

Let, $T(A) = [x, y, 1]$

$T(2A) = [x^2 - y^2, 2xy, 1]$

and $T(3A) = [x(x^2 - 3y^2), y(3x^2 - y^2), 1]$

$T(90^0 - 3A) = [y(3x^2 - y^2), x(x^2 - 3y^2), 1]$

Now $36^0 = 90^0 - 54^0$

$2A = 90^0 - 3A$ Thus $\sin 2A = \sin(90^0 - 3A)2xy = x(x^2 - 3y^2)$ i.e.
$2y = x^2 - 3y^2 = 1 - 4y^2$ $4y^2 + 2y - 1 = 0$.

because $x^2 + y^2 = 1$

$y = (-2 + 2\sqrt{5})/8 = (-1 + \sqrt{5})/4$

$$T(18^0) = [(\sqrt{10+2\sqrt5})/4, (-1+\sqrt5)/4, 1]$$

$$T(36^0) = [4+4\sqrt5, \sqrt{10-2\sqrt5}, 16]$$
$$= [1+\sqrt5/4, \sqrt{10-2\sqrt5}/4, 1]$$

Thus $\sin 18^0 = (\sqrt5 - 1)/4$, and $\cos 36^0 = (\sqrt5 + 1)/4$

Note that we need not remember any trigonometric formulae to find the desired result. This is the advantage of VM over usual methods.

EXERCISE :13.1

Ex. [1] If $\tan A = \sqrt{11}/5$, $\sec B = \sqrt{11}/7$, A lies in III quadrant and B lies in forth quadrant. Find the value of cosec A-sin B.

Ex. [2] If $\sin A/3 = \sin B/4 = 1/5$ and A and B both lie in second quadrant show that $4\cos A + 3\cos B = -5$.

Ex. [3] If $\tan = 2x(x+1)/(2x+1)$ obtain sin and cos .

Ex. [4] If $T(A) = [4, 3, 5]$ obtain $T(A\ 45^0)$

Ex. [5] In $\triangle ABC$, AB = AC and $T(B) = [12, 5, 13]$. obtain T (ABC)

Ex. [6] Find the value of $\cos 105^0 + \tan 135^0 - \sec 315^0$

Ex. [7] Show : $\cos A + \sin(270^0+A) - \sin(270^0-A) + \cos(180^0+A) = 0$.

Ex. [8] Show : $\cos 570^0 \sin 510^0 - \sin 330^0 \cos 390^0 = 0$.

Ex. [9] If $\tan\theta = 4/3$, $\sin\theta = 1/\sqrt2$, obtain the value of

$\tan(2\theta + 3\theta)$

Ex. [10] Show : $\sec^2 A(1 + \sec 2A) = 2\sec 2A$.

Ex. [11] Show : $\sin A \ \sin(60-A) \ \sin(60+A) = 1/4 \sin 3A$.

Ex. [12] Show : $\sin^2(\ /8 + A/2) - \sin^2(\ /8 - A/2) = 1/2$.

Ex. [13] Prove : $\cot A + \cot(60+A) + \cot(60-A) = 3\cot 3A$

Ex. [14] Prove : (I) $\tan 2A = 2\tan A / (1 - \tan^2 A)$

(ii) $\sin A = (2\tan A/2) / (1 + \tan^2 A/2)$

(iii) $\cos A = (1 - \tan^2 A/2) / (1 + \tan^2 A/2)$

Exercise 13.2: ***Derive the following identities :***

1) $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A$

2) $(\sin A + \cos A)(1 - \sin A \cos A) = \sin^3 A + \cos^3 A$

3) $(\sin A / 1 + \cos A) + (1 + \cos A / \sin A) = 2\operatorname{cosec} A$

4) $(\tan A / 1 - \cot A) + (\cot A / 1 - \tan A) = \sec A \operatorname{cosec} A + 1$

5) $(\sec A - \tan A / \sec A + \tan A) = 1 - 2\sec A \tan A + 2\tan^2 A$

6) $(1/\operatorname{cosec} A - \cot A) - (1/\sin A) = (1/\sin A) - (1/\operatorname{cosec} A + \cot A)$

7) $[1/(\sec^2 A - \cos^2 A) + 1/(\operatorname{cosec}^2 A - \sin^2 A)] =$

$(1 - \cos^2 A \sin^2 A) / (2 + \cos^2 A \sin^2 A)$

8) $[\sin(A-B) / \cos A \cos B] + [\sin(B-C) / \cos B \cos C]$

$+ [\sin(C-A) / \cos C \cos A] = 0$

9) $\cot(\pi/4 + \theta) \ \cot(\pi/4 - \theta) = 1$

10) $(1 - \cos A + \cos B - \cos(A+B) / (1 + \cos A - \cos B - \cos(A-) =$
$[\tan(A/2)\cot(B/2)$

11) Show : $(\sin\theta + \sin 2\theta) / (1 + \cos\theta + \cos 2\theta) = \tan\theta$

12) Prove :

$$(\sin A+\sin B)/(\sin A-\sin B)=\tan[(A+B)/2]/\tan[(A-B)/2]$$

13) Show : $(\sin^2 A-\sin^2 B)/(\sin A\cos A - \sin B\cos B)=\tan(A+B)$

14) Show : $\cos^3 2\theta + 3\cos 2\theta = 4(\cos^6\theta - \sin^6\theta)$

15) Show : $\cos^2 \alpha + \cos^2(\alpha + 120^0) + \cos^2(\alpha - 120^0) = 3/2$.

16) Solve the equations : (I) $\tan 2A = 4\tan A$

(ii) $1 - \sin A = 2\cos^2 A$

(iii) $3\cot 2A + 7\tan A = 5\operatorname{cosec} 2A$.

17) Show: $\cos^4 \pi/8 + \cos^4 3\pi/8 + \cos^4 5\pi/8 + \cos^4 7\pi/8 = 3/2$.

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CHAPTER 14.

INVERSE TRIGONOMETRIC FUNCTIONS

14.1 We know that if sinA = 1/2 then A = $\sin^{-1}(1/2)$. $\sin^{-1}(1/2)$ is inverse sine function. Similarly inverse cosine, inverse tan functions are defined. Here we shall assume that wherever needed, inverse of a function exists i.e. the given function is both one to one & onto. The required domain and co-domain are assumed. We will skip here the definition of an inverse function.

14.2 Using triplet method, the problems on inverse trigonometric functions can be solved easily. Here we present the solutions of some of them.

Illustrative Examples.

Ex. (1) Show that $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \ /4$.

Let $A = \tan^{-1}(1/2)$, tan A = 1/2 T(A) = [2 , 1 , 5]

$B = \tan^{-1}(1/3)$, tan B = 1/3 T(B) = [3 , 1 , 10]

T(A+B) = [5 , 5 , 50] tan(A+B) = 5/5 = 1 and A+B = / 4

$\tan^{-1}(1/2) + \tan^{-1}(1/3) = \ /4$.

Ex (2) Show : $\sin^{-1}(3/5) + \tan^{-1}2 = \tan^{-1}(-11/2)$.

Let $A = \sin^{-1}3/5$ sin A = 3/5 T(A) = [4 , 3 , 5]

$B = \tan^{-1}2$ tan B = 2 T(B) = [1 , 2 , 5]

$$T(A+B) = [\,-2\,,\ 11\,,\ 55\,]$$

$$\tan(A+B) = -11/2 \text{ and } A+B = \tan^{-1}(-11/2)$$

$$\tan^{-1}(\,3/5\,) + \tan^{-1} 2 = \tan^{-1}(-11/2\,).$$

Ex (3). Show : $3\tan^{-1}(\,1/2\,) = \tan^{-1}(\,11/2\,)$

Let, $A = \tan^{-1}(\,1/2\,)$, $\tan A = 1/2$ $T(A) = [\,2\,,\ 1\,,\ 5\,]$.

Now $T(2A) = [\,3\,,\ 4\,,\ 5\,]$ and $T(3A) = [\,2\,,\ 11\,,\ 55\,]$

$\tan 3A = 11/2$,and $3A = \tan^{-1}(\,11/2\,)$

$3\tan^{-1}(\,1/2\,) = \tan^{-1}(\,11/2\,)$.

Ex (4). Show : $\sin^{-1}(3/5) + \sin^{-1}(8/17) = \sin^{-1}(77/85)$

Let, $A = \sin^{-1} 3/5$, $\sin A = 3/5$ $T(A) = [\,4\,,\ 3\,,\ 5\,]$

$B = \sin^{-1}\ 8/17$, $\sin B = 8/17$ $T(B) = [\,15\,,\ 8\,,\ 17\,]$

Now $T(A+B) = [\,36\,,\ 77\,,\ 85\,]$

$\tan(A+B) = 77/85$ and $A + B = \tan^{-1}(77/85)$

$\sin^{-1} 3/5 + \sin^{-1} 8/17 = \tan^{-1} 77/85$.

Ex (5). Show : $\cos^{-1}(\,4/5\,) + \tan^{-1}(\,3/5\,) = \tan^{-1}(\,27/11)$.

Let, $A = \cos^{-1}\ 4/5$ $T(A) = [\,4\,,\ 3\,,\ 5\,]$

$B = \tan^{-1}\ 3/5$ $T(B) = [\,5\,,\ 3\,,\ 34\,]$

Now $T(A+B)=[11,\ 27,\ 729]$ then $\tan(A+B)=27/11$. etc.

Ex (6). Show $2\cos^{-1}(3/13) + \cot^{-1}(16/63) + (1/2)\cos^{-1}(7/25) =$.

Let, $A = \cos^{-1} 3/13$ $\cos A = 3/13$

$B = \cot^{-1} 16/63$ $\cot B = 16/63$

$C = \cos^{-1} 7/25$ $\qquad$ $\cot C = 7/25$

Thus T(A) = [3 , 2 , 13]and T(2A) = [5 , 12 , 13]

Now T(B) = [16 , 63 , 65]and T(C) = [7 , 24 , 25]

T(C/2) = [4 , 3 , 5]

T(2A + B + C/2) = [-1 , 0 , 1] = T() $\qquad$ 2A + B + (1/2)C =

Ex (7). Show : $(1/2) \tan^{-1}(12/5) = \tan^{-1}(2/3)$.

Let, $A = \tan^{-1} 12/5$ $\tan A = 12/5$

T(A) = [5 , 12 , 13] and T(A/2) = [18 , 12 , 613]

Thus $\tan A/2 = 12/18 = 2/3$ and $A/2 = \tan^{-1} 2/3$

$(1/2) \tan^{-1}(12/5) = \tan^{-1}(2/3)$.

Ex (8). Prove : $\tan^{-1}(1/4) + \tan^{-1}(2/9) = (1/2) \cos^{-1}(3/5)$.

Hint :-T(A) = [4, 1, 17], T(B) = [9, 2, 85], T(C) = [3, 4, 5]

T(A+B) = [2 , 1 , 5] and T(C/2) = [2 , 1 , 5] $\qquad$ etc.

Ex (9) Prove : $2 \tan^{-1}(1/3) + \tan^{-1}(1/7) = /4$

Hint : T(A) = [3, 1,10], T(2A) = [8, 6, 10],T(B) = [7,1,50]

T(2A+B) = [50 , 50 , 5000] = [1 , 1 , 2] etc.

Ex (10). Prove : $\tan^{-1}(1/3) + \tan^{-1}(1/5) + \tan^{-1}(1/7) + \tan^{-1}(1/8) = /4$

Hint : T(A) = [3 , 1 , 70], T(B) = [5 , 1 , 26]

T(C) = [7 , 1 , 50], T(D) = [8 , 1 , 65]

T(A+B) = [7 , 4 , 65] and T(C+D) = [11 , 3 , 130]

Thus T(A+B+C+D) = [1 , 1 , 2] etc.

EXERCISE :

Prove the following results

1) $\sin^{-1} 1/5 + \cot^{-1} 3 = /4$

2) $\cos^{-1} 4/5 + \cos^{-1} 12/13 = \cos^{-1} 33/65$.

3) $2\tan^{-1} 1/5 + \tan^{-1} 1/7 + 2\tan^{-1} 1/8 = /4$.

4) $\tan^{-1} x + \tan^{-1} (1-x)/(1+x) = /4$.

5) $2\cos^{-1} 3/13 + \cot^{-1} 16/63 + 1/2 \cos^{-1} 7/25 =$

6) $4\tan^{-1} 1/5 - \tan^{-1} 1/239 = /4$.

7) $\cos^{-1} 63/65 + 2\tan^{-1} 1/5 = \sin^{-1} 3/5$.

8) $\sin^{-1} 3/5 - \cos^{-1} 12/13 = \sin^{-1} 16/65$.

9) $3\tan^{-1} 1/4 + \tan^{-1} 1/20 = /4 - \tan^{-1} 1/1985$.

10) $\cos(2\tan^{-1} 1/7) = \sin(4\tan^{-1} 1/3)$

Hint :- Let, $A = \tan^{-1} 1/7$ T(A) = [7 , 1 , 50]

T(2A) = [24 , 7 , 25]

$\cos 2A = 24/25$

Let, $B = \tan^{-1} 1/3$ T(B) = [3 , 1 , 10]

T(2B) = [4 , 3 , 5]

T(4B) = [7 , 24 , 25]

$\sin 4B = 24/25$

$\cos 2A = \sin 4B$. etc.

11) $\sin^{-1} 1/5 + \cos^{-1} 3 = /4$.

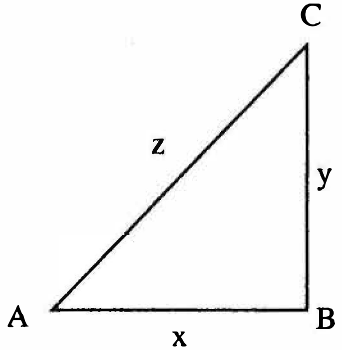
12) $\cos 9/82 + \text{cosec}^{-1} 41/4 = /4$.

CHAPTER 15.

HEIGHTS AND DISTANCES.

15.1 If x, y, z are the sides of a right-angled triangle (as shown in figure.) then we have the triplet of a angle A as

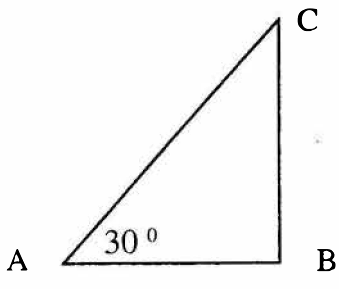
$$T(A) = [\ x\ ,\ y\ ,\ z\ ]$$



15.2 This definition is used to solve the problems on heights and distances by VM method. The angles mentioned in this chapter are measured in degrees.

Illustrative Examples.

1) A flagstaff is at a distance of 60 m from the observer on the road. The angle of elevation of flagstaff is 30^0.
Find its height.

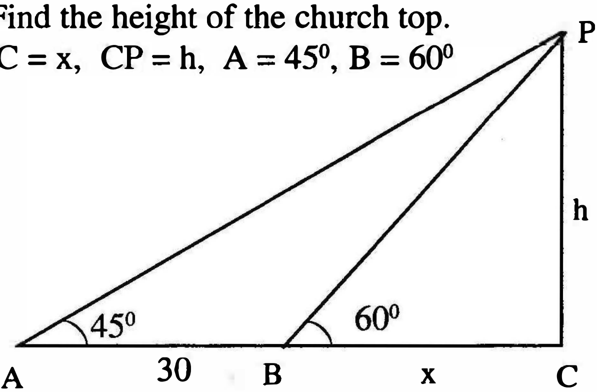


$T(30^0) = [3, 1, 2]$ $T(A) = [60, h, AB]$ comparing

$\sqrt{3}/60 = 1/h$ $h = 60/\sqrt{3} = 34.64$ m

2) The angles of elevation of a church top as seen from the two places, 30 m apart on the same line, are 45^0 and 60^0 respectively. Find the height of the church top.

Let, $BC = x$, $CP = h$, $A = 45^0$, $B = 60^0$



Now $T(A) = T(45)$ hence $[30+x, h, AP] = [1, 1, 2]$

$(30+x)/1 = (h/1)$ $\therefore h = 30 + x$ ..... (1)

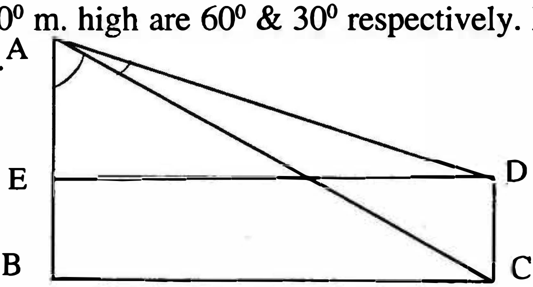
Also, $T(B) = T(60^0)$ hence $[x, h, PB] = [1, \sqrt{3}, 2]$

$\therefore (x/1) = (h/3)$ $x = h/\sqrt{3}$

$\therefore$ From (1) $h = 30 + (h/3)$

Thus $h = 30/(1-1/\sqrt{3})$ or $h = 70.98$ m.

3) The angles of depression of the foot and top of a tree, as seen from a tower 60^0 m. high are 60^0 & 30^0 respectively. Find the height of a tree.



Let, height of tree = h.
horizontal distance of tower from tree = x

$\angle EDA = 30^0$, $\angle BCA = 60^0$ T(EDA) = T(30^0)

[x , 60 - h , AD] = [$\sqrt{3}$, 1 , 2]

$\therefore$ ($\sqrt{3}$ / x) = 1 / (60 - h) $\therefore$ x = $\sqrt{3}$(60 - h) ..... (1)

Also, T(BCA) = T(60^0)

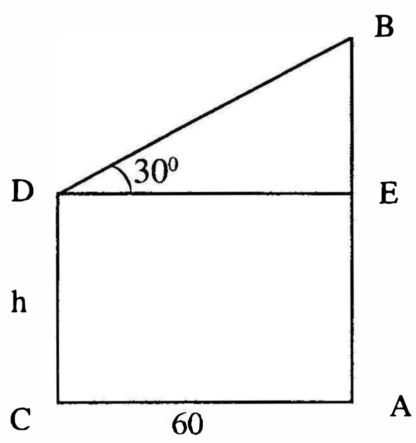
$\therefore$ [x , 60 , AC] = [1 , $\sqrt{3}$, 2]

x / 1 = 60 / $\sqrt{3}$ ..... (2)

From (1) & (2), 60 / $\sqrt{3}$ = $\sqrt{3}$(60 - h)

h = 40 m.

4) The angle of elevation of top of a 150 m height tower, as seen from another tower is 30^0, the horizontal distance between their bases standing on the same road is 60 m. Find the height of 2nd tower.

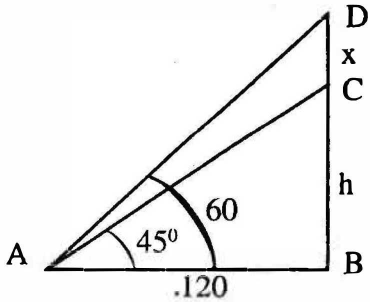


Let CD = h , $\angle EDB = 30^0$, T(EDB) = T(30^0)

[60 , 150 - h , DB] = [$\sqrt{3}$, 1 , 2] 60 / $\sqrt{3}$ = (150 - h) / 1

Thus h = $150 - 20\sqrt{3}$.

5) The angle of elevation of an unfinished tower from a point 120 m on the road is 45^0. After finish of the tower this angle becomes, 60^0. Find how much was the tower raised ?.



T(BAC) = T(45^0)   [120 , h , AC] = [1 , 1 ,$\sqrt{2}$]

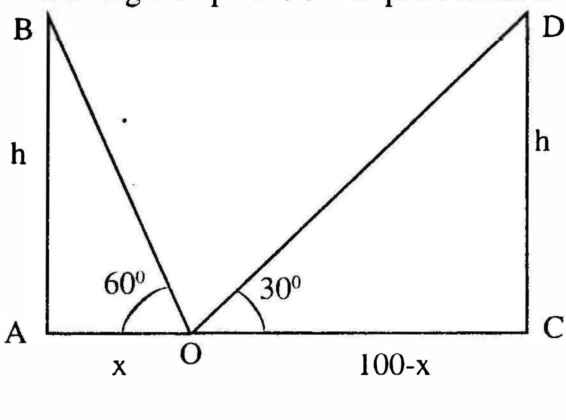
$\therefore$ 120 / 1 = h / 1   h = 120.

T (BAD) = T(60^0) [120 , h + x , AD] = [1, $\sqrt{3}$, 2]

$\therefore$ 120 / 1 = (h + x) / $\sqrt{3}$ h + x = 1203

i.e. x = $120\sqrt{3}$ - h = $120\sqrt{3}$ - 120 = 87.84 m.

6) The pillars of same height stand on the opposite sides of a road 100 m. wide. At a point on the road on the line joining their bases, the angles of elevation of there pillars are 60^0 & 30^0. Find the height of pillars & the position of the point.



T(AOB) = T(60^0) [x , h , OB] = [1, $\sqrt{3}$, 2]

$\therefore$ x / 1 = h / $\sqrt{3}$ ..... (1)

T(COD) = T(30^0) $\therefore$ [100 - x , h , OD] = [$\sqrt{3}$, 1 , 2]

$\therefore$ (100 - x)/$\sqrt{3}$ = h / 1 ..... (2)

From (1) & (2) (100 - x) /$\sqrt{3}$ = $\sqrt{3}$ x x=25m., AO=25 m

OC = 75 m. Also, h = x$\sqrt{3}$ = 25$\sqrt{3}$ = 43.3 m

EXERCISE :

1) At a certain point the angle of elevation of a tower is found to be such that its cotangent is (3 / 5), on walking 32 ft. directly towards the tower the angle of elevation is an angle whose cotangent is (2 / 5). Find the height of the tower.

2) At a point A , the angle of elevation of a tower is found to be such that its tangent is (5 / 12), on walking 240 ft. nearer the tower the tangent of angle of elevation is found to be (3 / 4), what is the height of the tower ?.

3) Find the height of a chimney when it is found that, on walking towards it 100 ft. in a horizontal line through its base, the angular elevation of its top changes from 30^0 to 45^0.

4) An observer on the top of a cliff, 200 ft. above the sea-level, observes the angles of depression of two ships at anchor to be 45^0 and 30^0 respectively. Find the distances between the ships if the line joining them points to the

base of the cliff.

5) The upper part of the tree broken over by the wind makes an angle of 30^0 with the ground, and the distance from the root to the point where the top of the tree touches the ground is 50 ft., what was the height of the tree ?.

6) What is the angle of elevation of the sun when length of the shadow of a pole is 3 times the height of the pole ?.

7) The shadow of a tower standing on a level plane is found to be 60 ft. longer when the sun's altitude is 30^0 than when it is 45^0. Prove that the height of the tower is $30(1+\sqrt{3})$ ft.

Answers :

11)	160 ft.	12)	225 ft.	13)	136.6 ft.
14)	146.4 ft.	15)	86.6 ft.	16)	115.35 ft.

❖ ❖ ❖

CHAPTER 16.

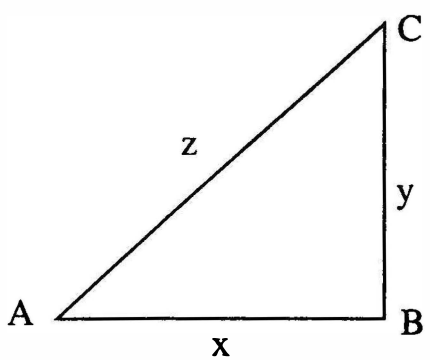
Solutions of Triangles .

16.1 A triangle has six elements : 3 sides and 3 angles. The process of finding all these elements if any two (or three) are given is called **solving a triangle**.

Here we discuss how VM is used to solve a triangle. In preceding chapters, we have given the concept of a triplet of an angle of a right-angled triangle, which we will use here.

16.2 Firstly we deal with a right .angled triangle.

Let its sides be x , y , z .



Then T(A) = [x, y, z],T(B)=[y, x, z] & T(90^0) = [0, 1, 1]

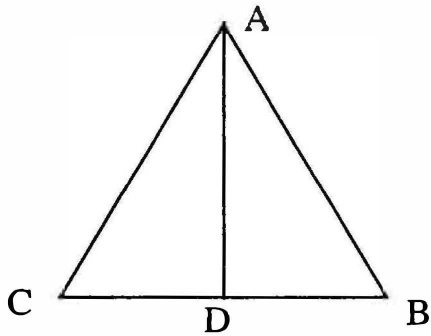
and we have sin A = y / z , sin B = x / z

Thus when all three sides are known, we can find the angles and thus all six elements are known. In a right-angled triangle it is not necessary to know all three sides, any two are enough.

We now proceed to the cases of triangle, which are not right angled. In all three cases the triplet of an angle is to be

found first.

16.3 Case [1] when all the three sides are given.



Let, BC = a, A = b and AB = c and all angles are acute.

Let, AD BC

To To find the triple of B, AD must be known.

Now, $AD^2 = AB^2 - BD^2 = AC^2 - DC^2$

$c^2 - BD^2 = b^2 - (a - BD)^2$

$2a\ BD = c^2 + a^2 - b^2$

$BD = (c^2 + a^2 - b^2) / 2a$

$AD^2 = c^2 - BD^2$

$T(B) = [\ BD, DA, AB\ ]$

$= [(c^2 + a^2 - b^2)/ 2a, DA, c\ ]$

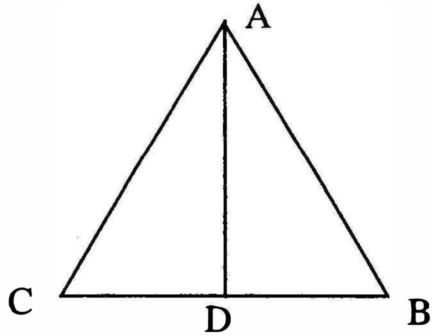
Similarly $T(C) = [\ DC, AD, AC\ ]$

$= [\ (a^2 + b^2 - c^2)/ 2a, AD, b\ ]$

and $\angle A = \pi - (B + C)$

Thus all angles are known.

Ex : Given a=6 , b=5 , c=4 obtain the angles of triangle ABC.



Here, BD $= (c^2 + a^2 - b^2)/2a = (16+36-25) / 12 = 9/4$

AD $= \sqrt{16^2 - (9/4)^2} = \sqrt{256 - 81/16} = 5\sqrt{7/4}$

T(B) $= [\,9/4 \ , \ 57/4 \ , \ 4\,]$

Hence $\tan B = 5\sqrt{7/9}$

Also, DC $= (a^2 + b^2 - c^2) / 2a = (36 + 25 - 16) / 12 = 15/4$

T(C) $= [\,15/4 \ , 57/4 \ , \ 5\,]$

$\tan C = \sqrt{7/3}$

and $\angle A = \pi - (B + C)$

Thus triangle is completely solved.

EXERCISE :

Solve the triangle completely from the following data.

A) $a = 32$, $b = 40$, $c = 66$ (ANS :- $C = 132^0$, 34 , 32)

B) $a = 56$, $b = 65$, $c = 33$ (ANS :- $C = 90^0$)

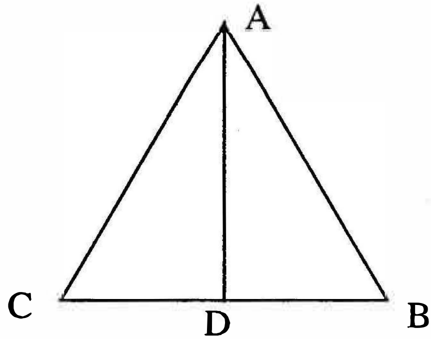
C) $a = 7$, $b = 4\sqrt{3}$, $c = 13$ (ANS :- $C = 30^0$)

D) $a = 2$, $b = \surd 6$, $c = 3 - 1$ (ANS :- 45^0 , 120^0 , 15^0)

E) $a = 2$, $b = \surd 6$, $c = 3 + 1$ (ANS :- 45^0 , 60^0 , 75^0)

16.4 Case [2]- Given two sides and the included angle.

Assume that sides a & c and the included angle B is given.



Let,T (B) = [BD , DA , AB] = [BD , AD , c]

Now BD = c. cos B . From this BD can be obtained.

T(C)=[CD, AD, AC]=[a-BD, AD, AC] & $AC^2= (a\text{-}BD)^2+AD^2$

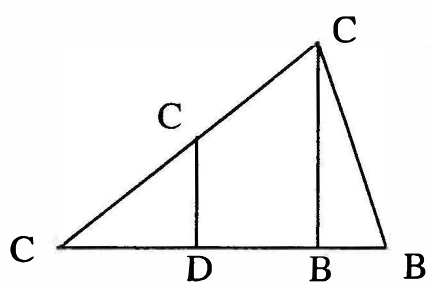
This will give angle C

Also, $BD = (c^2 + a^2 - b^2) / 2a$. Thi swill give b

Hence triangle is completely solved.

Ex : Given T(B) = [3 , 4 , 5], A BC = 21 , AB = 10

find all the angles and all sides of triangle ABC.



At E be midpoint of AB.

Draw perpendiculars ED and AF.

BE = 5

Now, BD = BE cos B = 5 (3 / 5) = 3 ED = 4

As BDE & BFA are similar. AF / ED = AB / AE .

Thus (AF / 4) = (10 / 5) AF = 8

Also, BC = 21 and BD = 3 FC = 18

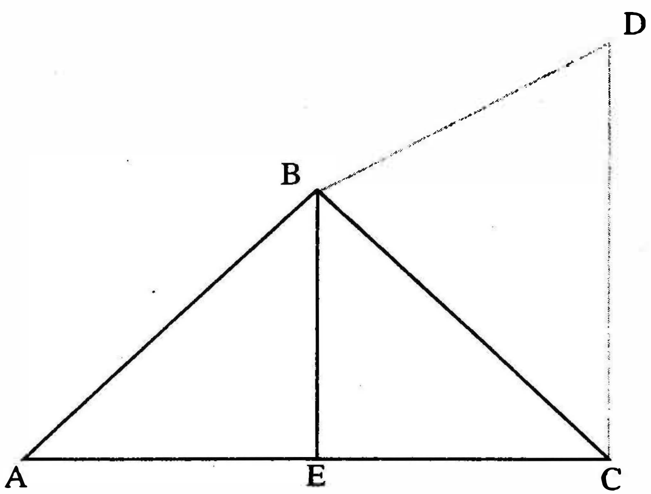
Now $AC^2 = FC^2 + AF^2 = 18^2 + 8^2$

AC = 388 = b

T(C) = [18 , 8 , 388]

and A = - (B + C)

Ex : Given b = 3 , c = 1, A = 30^0 solve the triangle.



Here $A = 30^0$, $T(A) = [\sqrt{3}, 1, 2]$

Draw BE to AC. Exlend AB upto D so that $ACD = 90^0$

As $(AD / AB) = (CD / BE)$, $BE = 1 / 2$

Also, $(AE / AC) = (AB / AD)$, $AC = 3 / 2$ CE $=3/2$

Now, $BC^2 = CE^2 + BE^2 = (3/2)^2 + (1/2)^2 = 1$

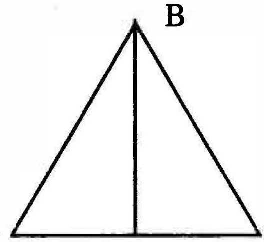
$T(C) = [CE, BE, BC] = [3/2, 1/2, 1] = [3, 1, 2]$

$C = 30^0$ and $B = 180 - (A + C) = 120^0$

Thus $A = 30^0$, $a = 1$, $B = 120^0$, $b = 3$, $C=30^0$, $c = 1$

Hence triangle is solved.

Ex : Given $a = 2$, $b = 1 + 3$, $\angle C = 60^0$, solve the triangle.



Now, $T(C) = T(60^0) = [1, \sqrt{3}, 2]$

$CD = 1$, $DB = \sqrt{3}$. $CB = 2$

And $CD = 1 + \sqrt{3}$ $AD = \sqrt{3}$

Here $AB = \sqrt{(AD^2 + DB^2)} = \sqrt{6}$

$T(A) = [\sqrt{3}, \sqrt{3}, AB] = [\sqrt{3}, \sqrt{3}, 6] = [1, 1, \sqrt{2}]$

$A = 45^0$ and $C = 60^0$, $B = 180 - (45 + 60) = 75^0$

The triangle is solved.

EXERCISE :

Solve the triangle in the following cases.

[1] $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, $\angle C = 60^0$

[2] $b = 1$, $c = \sqrt{3} - 1$, $\angle A = 60^0$, Find a

[3] $b = 91$, $c = 125$, $\tan A / 2 = 17 / 6$, Show that $a = 204$

[4] $a = 5$, $b = 4$, $\cos(A - B) = 31/32$ then show that $c = 6$

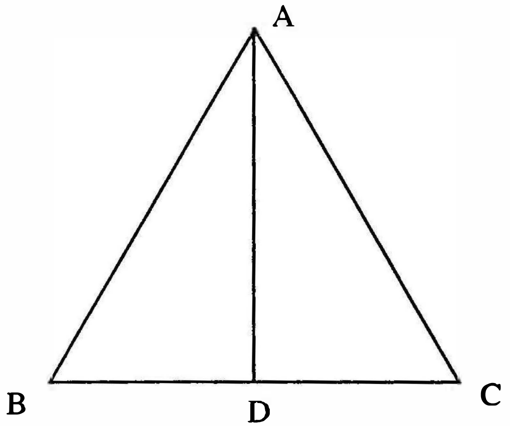
[5] $a = 40$, $b = 40\sqrt{3}$, $C = 30^0$

Answers :-

[1] $\sqrt{6}$, 15^0 , 105^0 **[2]** $a = .8965$ **[5]** 40 , 120^0 , 30^0.

16.5 SINE RULE :

To prove the sine rule: $(a / \sin A) = (b / \sin B) = (c / \sin C)$



Proof :

$T(B) = [\ BD , AD , AB\ ]$ and $T(C) = [\ CD , AD , AC\ ]$

$\sin B = AD/AB = AD / c$ and $\sin C = AD / AC = AD / b$

$AD = b \sin C = c \sin B$

$b / \sin B = c / \sin C$ etc.

16.6 Case [3] - Two angles and a side is given.

Ex : Given T(B) = [12 , 5 , 13], T(C) = [3 , 4 , 5], b = 20

Solve the triangle.

sin B = 5 / 13, sin C = 4 / 5

and from b sin C = c sin B we get 20 (4 / 5) = c (5 / 13) = 41.6

Now T(B + C) = [16 , 63 , 65] and T(180) = [-1 , 0 , 1]

Hence T(A) = T(180 - B + C) = [-16 , -63 , 65]

Thus sin A = -63 / 65

Now, a sin B = b sin A i.e. a (5 / 13) = 20 (-63 / 65)

$a = \left| -252/5 \right| = 252/5$

Thus a = 252 / 5 sin A = -63 / 65

b = 20 sin B = 5 / 13

c = 41.6 sin C = 4 / 5

Ex : Given b = 2 , c = 6 , T(C) = [3 , 4 , 5].Find T(B).

Let, T(B) = [x , y , z]

Now b sin C = c sin B hence 2 (4 / 5) = 6 (y / z)

- y / z = 4 / 15

Thus T (B) = [x , 4 , 15]

Now $x^2 = z^2 - y^2 = 225 - 16 = 209$

$T(B) = [\sqrt{209}, 4, 15]$

EXERCISE : Solve the triangles :

1) a = 2, $c = \sqrt{3} + 1$, $A = 45^0$

2) a = 100 , $c = 100\sqrt{2}$, $A = 30^0$

3) a = 6, b = 6, $A = 30^0$

ANSWERS :

1) $B = 30^0$, $C = 105^0$, $b = \sqrt{2}$

2) $B = 15^0$, $C = 135^0$, b = 51.76

3) $C = 4\sqrt{3} \pm 2\sqrt{5}$

4) $b = 50\sqrt{3}$, $c = 150^0$, $B = 30^0$ 4) Rt. angled triangle, $a = 100\sqrt{3}$

5) $2b = 3a$ and $\tan A = \sqrt{3} / 5$ 5) Rt. angled triangle.

Note :- In each case, we will get two triangles of which answer for one is given here.

16.7 The three angles ABC are given.

In this case it is convenient to use the sine rule.

$a / \sin A = b / \sin B = c / \sin C$

Ex : Given $A = 45^0$, $B = 75^0$, $C = 60^0$. Solve the triangle.

We have $T(A) = [\ 1 , 1 , 2\ ]$

$T(B) = [3\text{-}1 , 3\text{+}1 , 22\ ]$

$T(C) = [\ 1 , 3 , 2\ ]$

By sine rule : $a / (1 / 2) = b / (3+1 / 2) = c / (3 / 2)$ etc.

EXERCISE :

1) If $\cos A = 17 / 22$, $\cos C = 1 / 14$. Obtain the ratio $a : b : c$

2) The angles of a triangle are in the ratio 1 : 2 : 7.Obtain sides.

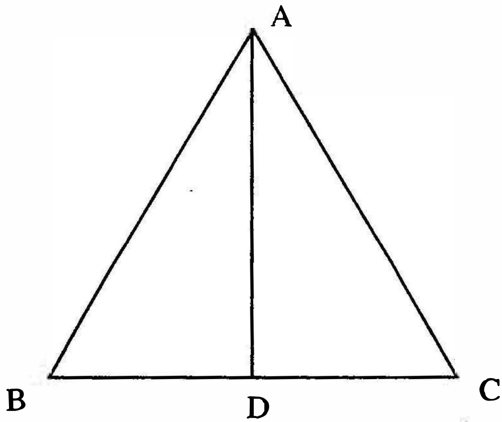
The advantage of VM over traditional Math's is that we need to remember the triples only.

16.8 The cosine of angle A is given by

$\cos A = b^2 + c^2 - a^2 / 2bc.$

Definition :- $b^2 + c^2 - a^2$ is called the angle deficiency for angle A. It is useful to find the angles if sides are given.

To prove : In triangle ABC show that $a = b \cos C + c \cos B$



Here $T(B) = [\ BD\ ,\ DA\ ,\ AB\ ]$ and $T(C) = [\ CD\ ,\ DA\ ,\ AC\ ]$

$$b \cos C + c \cos B = AC(CD / AC) + AB\ (BD / AB)$$

$$= CD + BD = BC = a.$$

The other results required for solving a triangle are given below & can be easily verified.

1) $\tan [(\ B - C\ ) / 2] = \cot (\ A / 2\ )\ (b - c\ ) / (\ b + c\ )$

2) $\cos A = (\ b^2 + c^2 - a^2\ ) / 2\ c\ a$

3) $\tan (\ A / 2\ ) = (s-b)(s-c) / s(s-a)$ where $2s = a + b + c$

4) $\sin A = (\ 2 / bc\ )\ \ s\ (\ s - a)\ (\ s - b\ )\ (\ s - c\ )$

Ex : If $a = 13$, $b = 14$, $c = 15$, obtain $\tan (B / 2)$ and $\sin (A / 2)$

Here $s = (\ a + b + c\ ) / 2 = (\ 13 + 14 + 15\ ) / 2 = 21$

$\sin (A / 2) = [(s-b)(s-c) / bc] = [\ (7\ \ 6) / (14\ \ 15)\ ] = 1 / 5$

$\tan (A / 2) = [(s-b)(s-c) / s(b-a)] = [\ (6\ \ 8) / (21\ \ 7)\ ] = 4 / 7$

PART (III)

COORDINATE GEOMETRY

CHAPTER 17.

STRAIGHT LINES

Sutra {1} उर्ध्व तिर्यग् भ्याम् Urdhva tiryakbhyam

(Vertically and crosswise)

17.1 Of all the quantities in coordinate Geometry, the equation of a straight line finds a premier place. Hence we shall begin with equation to a straight line.

The equation a line passing through two points (x_1, y_1) and (x_2, y_2) is

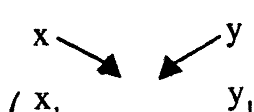
$$y - y_1 = [(y_2 - y_1) / (x_2 - x_1)] (x - x_1)$$

17.2 The method of VM gives a simple way of obtaining this result. Here we use the sutra {1} as follows.

Coeff of y is $(x_1 - x_2)$

Coeff of x is $(y_1 - y_2)$

Constant is $(x_1y_2 - y_1x_2)$



equation to line is $(x_1 - x_2) y = (y_1-y_2) x +(x_1y_2-y_1x_2)$

Ex 1) Find the equation to a line through the points (6 , 5) and (4 , 1).

equation to a line is

x	y
6	5
4	1

$(6 - 4)\ y = (5 - 1)\ x + (61 - 45)$

$2y \quad = 4x - 14$ is $y = 2x - 7$

Ex 2) Find the equation to the line through origin and the point (3, 4)

$y\ (3 - 0) = x\ (4 - 0) + 0$

$3y = 4x$

x	y
0	0
3	4

Exercise :

Obtain the equation to a line passing through the following pair of points.

(i) (7 , 3) , (2 , 1) (ii) (-3 , 0) , (1 , 6) (iii) (8, -1), (2, -5)

(iv) (-1, 7) , (-5 ,4) (v) (0 , -3) , (-1, -2)

ANSWER :

(i) $5y = 2x + 1$ (ii) $2y = 3x + 9$ (iii) $3y = 2x - 19$

(iv) $4y = 3x + 31$ (v) $y = -x - 3$

17.3 Triplet for a line :

The position of a line in two dimensional coordinate geometry is known by its slopem. Consider a line $y = m\ x + c$

Or $\quad x / 1 = (y - c) / m.$

Definition :

The triplet of line $x / a = y\text{-}c / b$ is $(a , b , \sqrt{a^2+b^2})$

Note:

(i) Two lines with triplets (x_1 , y_1 , z_1) and (x_2 , y_2 , z) are parallel if $x_1 / x_2 = y_1 / y_2$.

(ii) These lines are perpendicular if $x_1x_2 + y_1y_2 = 0$.

(iii) The triplet of X-axis is $[1 , 0 , 1]$.

The triplet of Y-axis is $[0 , 1 , 1]$.

(iv) These triplets coincide with those of angle 0^0 and 90^0 respectively.

(v) The triplet of a line is same as the triplet of the angle that the line makes with X-axis.

17.4 Angle between the lines.

To find the angle between the lines $a_1x + b_1y + c_1 = 0$ &

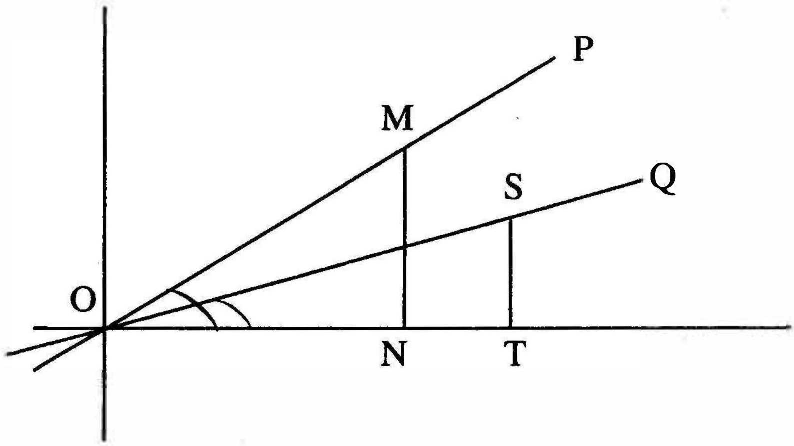
$a_2x + b_2y + c_2 = 0.$

The line $a_1x + b_1y + c_1 = 0$ is put as, $x / b_1 = y /(-a_1) + c_1 / (- a_1b_1)$.

$\therefore$ Triplet of line $\quad a_1x + b_1y + c_1 = 0$ is $[b_1, -a_1 , -]$.

Similarly Triplet of $a_2x + b_2y + c_2 = 0$ is $[b_2, -a_2 , -]$.

Let OP , OQ represent these two lines.



Let $\angle A = \angle NOM$, $\angle B = \angle TOS$

then SOM = A - B is the angle between the lines.

Triplet of line OP : $[b_1 , -a_1 , -]$ = Triplet of $\angle A$.

Triplet of line OS : $[b_2 , -a_2 , -]$ = Triplet of $\angle B$.

$\therefore$ Triplet of A - B = $[b_1b_2 + a_1a_2 , b_1a_2 - a_1b_2 , -]$

$\therefore \tan (A - B) = | (b_1a_2 - a_1b_2) / (b_1b_2 + a_1a_2) |$.

This gives angle between two lines.

This coincides with the formula $\tan(A-B)=(m-m_2)/(1+m_1m_2)$

where

$m_1$and m_2 are slopes of the line.

Ex : Find the angle between the lines 2x+3y=4 and 3x+4y = 5.

Solution : Triplet of line (I) = [3 , -2 , -]

Triplet of line (II) = [4 , -3 , -]

Triplet of the angle between the lines = [(34)+(23),(-42)+(33),-]

$= [\ 18\ ,\ 1\ ,\ -\ ]$

$$\boxed{\therefore\ \tan\theta = 1/18.}$$

17.5 Lines in 3 - Dimensions

Now we consider the lines in 3-d geometry. In this case, we define a quadruplet.

For a three dimensional geometry, equation to a line passing through a point (α, β, γ) and with direction rations l , m , n is

$$(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n.$$

Definition : The quadruplet of the line x/l= y/m= z / n is

$Q(\text{ line }) = (l\ ,\ m\ ,\ n\ ,\ \sqrt{l^2 + m^2 + n^2})$

To find the angles between the lines:

$(x - \alpha_1)/l_1 = (y - \beta_1)/m_1 = (z - \gamma_1)/n_1$ (I)

$(x - \alpha_2)/l_2 = (y - \beta_2)/m_2 = (z - \gamma_2)/n_2.$

(II)

We have $Q(\text{ line I }) = (l_1\ ,\ m_1\ ,\ n_1\ ,\ r_1)$

$Q(\text{ line II }) = (l_2\ ,\ m_2\ ,\ n_2\ ,\ r_2)$

Where $r_1^{\,2} = l_1^{\,2} + m_1^{\,2} + n_1^{\,2}$

$r_2^{\,2} = l_2^{\,2} + m_2^{\,2} + n_2^{\,2}.$

If is angle between the lines

Then, $T(\theta) = [\ l_1l_2 + m_1m_2 + n_1n_2\ ,\ —\ ,\ r_1r_2\ ]$

$$\cos\theta = \frac{l_1l_2 + m_1m_2 + n_1n_2}{r_1r_2}$$

We use Sutra {1} for this result.

The proof of the above result is similar to that for lines in two dimensions and hence left to reader.

Note : If $\cos\theta = 0$, then lines are perpendicular and $\sin\theta = 0$, then lines are parallel.

Ex : Find the angle between the lines.

$(x-1)/2 = (y-2)/3 = (z-3)/4$ and

$(x-2)/3 = (y-3)/4 = (z-4)/5$

Here $Q(\text{line I}) = (2, 3, 4, \sqrt{29})$

$Q(\text{line II}) = (3, 4, 5, \sqrt{50})$

$T(\theta) = [\,(23) + (34) + (45)\ ,\ \text{—}\ ,\ \sqrt{29} \times \sqrt{50}\,]$

$= [\,38, \text{—}, \sqrt{29} \times \sqrt{50}\,]$

$\cos\theta = 38 / \sqrt{29 \times 50}.$

Ex : Show that the lines $x/1 = y/3 = z/3$ and

$(x-1)/3 = (y-2)/2 = (z-5)/-3$ are perpendicular.

$Q(\text{line I}) = (1, 3, 4, 19)$

$Q(\text{line II}) = (3, 2, -3, 22)$

$T(\theta) = [\,(13) + (32) + (3(-3))\ ,\ \text{—}\ ,\ \sqrt{19 \times 22}\,]$

$= [\,0\ ,\ \text{—}\ ,\ \sqrt{19 \times 22}\,]$

$\therefore \tan\theta \longrightarrow \infty$, hence lines are perpendicular.

Ex : Show that the lines $2x - 3y + 4 = 0$ and

$3x + 2y + 5 = 0$ are perpendicular.

Triplet of first line : $[-3, -2, -]$

Triplet of second line : $[2, -3, -]$

$\therefore$Triplet of angle between these lines$=[(-32)+(-2)(-3),(- 4-9),-]$

$= [0, -13, -]$

$= [0, 1, -]$

$= T(90^0)$

Hence lines are perpendicular.

Note :- Lines are perpendicular if the triplet of the angle between them is

$[0, 1, -]$ and are parallel if $T(\theta) = [1, 0, —]$.

Ex : Show that the lines $4x + 5y + 7 = 0$ and

$8x + 10y + 11 = 0$ are parallel.

T(line I) $= (5, -4, -)$

T(line II) $= (10, -8, -)$

T(angle between the lines) $= (82, 0, -)$

$= (1, 0, -)$

$= T(0^0)$

$\therefore$ lines are parallel.

EXERCISE :

1) Find the angle between the lines :

(i). $x + y + 1 = 0$ and $2x - y + 3 = 0$

(ii). $3x + 2y + 4 = 0$ and $4x + 3y + 2 = 0$

(iii). $x + y = 0$ and $x - y = 0$

2) Show that lines given below are perpendicular.

(i). $y = (2 / 3)x + 7$ and $3y + 2x + 7 = 0$

(ii). $x + y + 1 = 0$ and $x - y - 1 = 0$

(iii). $2x + 5y + 9 = 0$ and $5x - 2y - 9 = 0$

3) Find the angle between the lines :

(i). x/2 = y/-1 = z-1/1 & x-1/3 = y+2/1 = z/3

(ii). x/1 = y/2 = z/3 & x/4 = y/5 = z/6

(iii). x-1/3 = y-4/2 = z+1/1 & x+4/2=y/3=z-11/-12

17.6 Length of a perpendicular

To find the length of perpendicular from a point, (x_1, y_1) on the line

$ax + by + c = 0.$

Line $ax + by + c = 0$ can be put as $x/-b = (y+c/b)/a$.

Now shift the origin to $(0, -c/b)$. So that this line will

pass through the origin.

Also the coordinates of point (x_1 , y_1) will become $(x_1 , 0_1 , y_1 -(-c/b))$, i.e. $(x_1 , y_1 , +c/b)$.

Now, $\quad T(\text{line}) = [\, b , -a , \sqrt{(a^2 + b^2)}\,]$

$\quad T(\text{line}) = [\, x_1 , y_1 + c/b , op\,]$

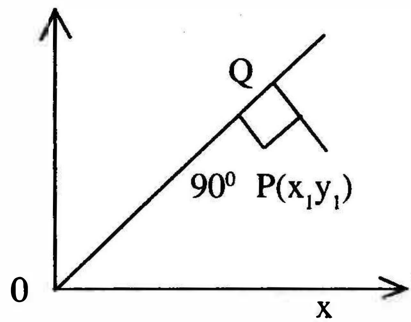
Now subtract the angles, we will get the triplet of the triangle POQ.

$$T(\text{angle } POQ) = [\, — , ax_1 + b(y_1 + c/b) , OP \ \sqrt{(a^2 + b^2)}\,]$$

$$= [\, — , ax_1 + by_1 + c ; OP , \sqrt{(a^2 + b^2)}\,] \qquad y$$

$$= [\, — , ax_1 + by_1 + c / \sqrt{(a^2 + b^2)}, OP\,]$$

$$= [\, OQ , PQ , OP\,]$$



Thus $PQ = (ax_1 + by_1 + c) / \sqrt{(a^2 + b^2)}$

This formula agrees with the traditional formula for finding length of perpendicular.

Ex : Find the length of perpendicular from a point of (3 , 2) on the line

$$y = 4x.$$

Solution : The line is put as $x/1 = y/4$

T(line) $= (1 , 4 , \sqrt{17})$

T(point) $= (2 , 3 , \sqrt{13})$

T(angle that OP makes with line) $= (14, 42 - 31, \sqrt{13} \times \sqrt{17})$

$= (14, 5, \sqrt{13} \times \sqrt{17})$

$= (14/\sqrt{17}, 5/\sqrt{17}, \sqrt{13})$

Length of perpendicular $= 5/\sqrt{17}$.

Verification $= p = (ax_1 + by_1 + c) / \sqrt{a^2 + b^2}$

$= 4(2) - 1(3) + 0 / \sqrt{4^2 + 1^2}$

$= 5 / \sqrt{17}$

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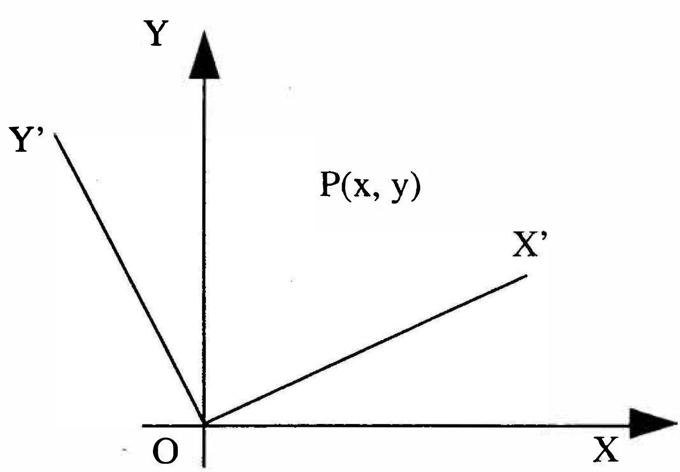
CHAPTER 18.

TRANSFORMATIONS IN A PLANE

18.1 Let a point **P** have coordinates (x , y) w.r.t. O as origin. If the origin is shifted to a point (h , k) the new coordinates of **P** are (x - h , y - k). In another case if the origin is fixed, but the axes are rotated anticlockwise through angle θ then the new coordinates of **P** are(X, Y) where.



$X = x \cos \theta - y \sin \theta$ And $Y = x \sin \theta + y \cos \theta$.

The proofs of these results are omitted for brevity.

18.2 With help of VM using triples, this orientation can be formulised very easily. The procedure is simple : Write the triple of sum of that for the point and the angle. Few solved examples will reveal the procedure.

Here, the triplet of a point **P** is the triplet of the angle XOP, all angles are measured in degrees.

Ex 1). Find the new coordinates of point **P** (7 , 3) when axes are rotated through 90^0.

We have : T (P) = [7, 3, $\sqrt{58}$] , T (90^0) = [0 , 1 , 1]

Let **P** change to **P**^{1} after rotation.

Then T (**P**^{1}) = T (P + 90) = [-3 , 7 , $\sqrt{58}$].

[We are not interested in the last element of the triplet [x , y , z] as $z^2 = x^2 + y^2$]

Coordinates of P (X , Y) = (- 3 , 7).

In VM we need to remember the results of T (A + B) and T (A - B) only.

Ex 2). **P** has coordinates (3 , 2). Obtain its new coordinates if axes are rotated anticlockwise through 45^0, keeping origin fixed.

Here T (P) = [3 , 2 , —] , T (45^0) = [1 , 1 , —]

T (P^1) = T (P + 45^0) = [3-2 , 3+2 , —] = [1 , 5 , —]

$\therefore$ **P** has coordinates (1 , 5)

But using formula X = x cos θ - y sin θ,

Y = x sin θ + y cos θ

we get X = (1/2 , 5/2)

Why this ambiguity ?

The answer is simple : we have $\cos^2\theta + \sin^2\theta = 1$ & we assume that

x = cos θ , y = sin θ.

Hence the z element of triplet of θ should be made unity. Hence we solve the above problem again :

Here $T(P) = [3, 2, —]$, $T(45^0) = [1, 1, 2]$ is put as

$T(45^0) = [1/2 \ , \ 1/2 \ , \ 1]$

Hence $T(P) = T(P + 45^0) = [1/\sqrt{2} \ , \ 5/\sqrt{2} \ , \ —]$

Thus Coordinates of **P** = $(1/\sqrt{2} \ , \ 5/\sqrt{2})$.

Ex 3) Find the new coordinates of a point **P** (5 , 2) when the axes are rotated in positive direction through an angle, whose triplet is [4 , 3 , 5].

Here $T(P) = [5, 2, \sqrt{29}]$, $T(\theta) = [4, 3, 5]$

Hence $T(P) = T(P + \theta) = [14 \ , \ 23 \ , \ 5\sqrt{29}]$

Thus coordinates **P** should be (14 , 23). But it is not so. The reason is that the length **OP** does not change due to orientation.

Now the z element of $T(P)$ is $\sqrt{29}$

$OP = \sqrt{29}$ $\qquad$ $OP = \sqrt{29}$.

and z element of $T(P)$ is $5\sqrt{29}$. Hence we divide $T(P)$ by 5.

$T(P) = (14/5 \ , \ 23/5, \sqrt{29})$

Thus new coordinates of P are (14/5 , 23/5).

Ex 3). Find the new coordinates of **P** (4 , 7) when the axes are rotated in positive direction about angle θ with triplet

[3 , 4 , 5] and origin is shifted to point [2 , 3].

Coordinates of are P (4 , 7). New origin is (2 , 3). Hence new coordinates of P are P (4 - 2 , 7 - 3) i.e. P (2 , 4).

Now $T(P) = [2 \ , \ 4 \ , \sqrt{20}]$, $T(A) = [3 \ , \ 4 \ , \ 5]$

Hence $T(\mathbf{P}^1) = T(P+A) = [-10, \sqrt{20}, \ 520] = [-2, 4, \sqrt{20}]$

Now shift back to old origin

$T(P^1) = [-2+2\ ,\ 4+3\ ,\ —] = [0\ ,\ 7\ ,\ —]$

$\therefore P^1 = (0\ ,\ 7)$

Ex 4). Rotate P (8 , - 5) through 135^0 about the new origin o1 (7 , 7).

P = (8 , - 5) changes to (8-7 , -5-7) = (4 , -12) w.r.t. new origin O

Now $T(P) = [1\ ,\ -12, \sqrt{145}]$, $T(135^0) = [-1, 1, \sqrt{2}]$

$T(P) = T(P + 135^0) = [11,\ 13, \sqrt{290}]$

$= [11/\sqrt{2}\ ,\ 13/\sqrt{2}\ ,\ \sqrt{145}]$

(because OP = OP)

$P = (11/2 + 7, 13/\sqrt{2} + 7)$ w.r.t. O.

Exercise :

In following examples, obtain the new coordinates of **P** when the axes are rotated, in positive direction through about the new origin O^1.

1). P = (2 , 3), $T(\theta) = [-4, 35]$ $O^1 = O = (0, 0)$.

2). P = (1 , 7), $\theta = 60^0$, $O^1 = (3, 5)$.

3). P = (-1 , -2), $\theta = 120^0$, $O^1 = (-2, 5)$.

4). P = (4 , -3), $\theta = 30^0$, $O^1 = (-3, -4)$.

5). P = (6 , 2), $T(\theta) = [2\ ,\ 1\ ,5]$, $O^1 = [1, 1]$.

6). P = (-2 , 5), $\theta = 270^0$, O^1 is $O = (0, 0)$.

18.3 Now we shall see the effect of rotation of axes on the equation of line, curve etc.

Ex 1). Find the new equation of a line y = 2x + 1 when axes are rotated through angle where $T(\theta) = [4, 3, 5]$ and the

origin is fixed.

Let, P (x , y) be any point on the line.

Let, new coordinates of P, after effect of rotation be P(x ,y).

Now T (P) = (X , Y , Z) where $Z^2 = X^2 + Y^2$

T (θ) = [4 , 3 , 5]

T ($\mathbf{P}^1$ - θ) = [4X + 3Y , 3X - 4Y , 5Z]

= [4X + 3Y / 5 , -3X + 4Y / 5 , Z]

= [x , y , z] , say.

x = 4X + 3Y / 5 , y = -3X + 4Y / 5

Put in y = 2x + 1 4Y - 3X / 5 = 2(4X + 3Y / 5) + 1

i.e. 2Y - 11X + 5 = 0.

Ex 2). Rotate the line 2x + 3y = 4 through angle where T (θ) = [3 , 4 , 5]

Let, after rotation, any point P (x , y) on the line be changed to a point P (x, y).

Let T (P) = [X , Y , Z]

& T (θ) = [3 , 4 , 5]

T ($\mathbf{P}^1$ - θ) = [3X + 4Y , 3Y - 4X , 5Z]

= [3X + 4Y / 5, 3Y - 4X / 5, Z

= [x , y , z]

Put x = 3X + 4Y / 5 , y = 3Y - 4X / 5

in 2x + 3y = 4.

2 (3X + 4Y / 5) + 3 (3Y - 4X / 5) = 4

6X - 17Y + 20 = 0.

Ex 3). Find the new equation to parabola $y = x^2$ when axes are rotated through 90^0 clockwise.

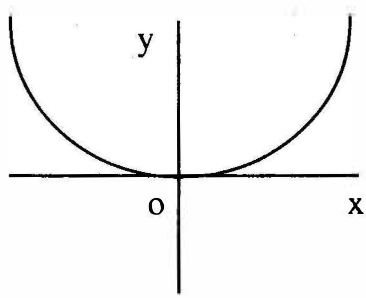
As above,

$$T(P) = [X \; , \; Y \; , \; Z]$$

$$T(90^0) = [0 \; , \; 1 \; , \; 1]$$

$\therefore$ T (P+90) = [-Y, X , Z]= Here T (P+90) instead of T (P - 90).

$$= [x \; , \; y \; , \; z]$$



$\therefore$ Put $x = -y$ $\quad\quad$ $y = x$

We get $X = (-Y)^2$

$$Y^2 = X$$

new form

Ex : Find the new form of the ellipse $x^2 / 4 + y^2 / 9 = 1$, when axes are rotated through 45^0, in positive direction. (solution is left to the reader).

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PART (I V) — CALCULUS

Chapter 19

DERIVATIVES

19.1 The concept of a derivative of a function originates from the limiting value of a ratio. Since VM does not deal with the limit of a function we will not define a derivative. On the other hand VM is useful in application of its formulae. Here we study the methods of differentiation of product of two functions with assumption that reader is familiar with derivatives of different functions and basic rules of differentiation.

Sutra : ऊर्ध्वतिर्यग्भ्याम् Urdhva tiryakbhyam

(Vertically and crosswise)

There is striking similarity between method of finding product of numbers by above sutra and method of finding derivative of product of two or three functions. Let us first revise the method of multiplication by above sutra.

Ex 1. Multiply 35 x 27

Ans : We write :

35	Note : For the central
X 27	part we have
------------------------	
6 | 31 | 35	(3 x 7)+(2 x 5) = 31.
= 945	

Ex 2. Multiply 246 x ***531***

Ans : We write :

246	<u>Note :</u> For the central
X 531	part we write
------------------------	
10 | 26 | 44 | 22 | 06	(2x1)+(4x3)+(5x6) = 44
= 130626.	

Ex 3. Multiply 1345 x ***6802***

Ans:We write : 1 3 4 5	<u>Note</u> : For the central
X 6 8 0 2	part we have
----------------------------	
06 26 48 64 46 08 10	
	(1x2)+(3x0)+(4x8)+(3x0)+(6x5)=64.
= 9148690	

Ex 4. Multiply 35 27 41

Ans : We write :

35	<u>Note</u> : For part II
X 27	from left we get
41	(5x2x4)+(7x34x1)+(1x2x3)=130
------------------------------------	
24 130 171 35	
= 38745	

19.2 Term wise Coefficients :

The coefficients of terms in first and successive derivatives of product of two functions are as follows.

1. For first order derivatives : ${}^1c_0 , {}^1c_0$ i.e. 1 , 1.
2. For second order derivatives : ${}^2c_0 , {}^2c_1 , {}^2c_2$ i.e. 1 , 2 ; 1.
3. For third order derivatives : ${}^3c_0 , {}^3c_1 , {}^3c_2 , {}^3c_3$
   i.e. 1, 3, 3, 1.
4. For forth order derivatives : ${}^4c_0 , {}^4c_1 , {}^4c_2 , {}^4c_3 , {}^4c_4$
   i.e. 1, 4, 6, 4, 1.

Thus,

5. For nth order derivative the coefficients are

$${}^nc_0 , {}^nc_1 , {}^nc_2 , {}^nc_3 , {}^nc_4 \text{ ------------------ } {}^nc_n.$$

$$\text{Where } {}^nc_r = \frac{n!}{r!(n-r)!}$$

19.3 The structures for derivatives of product of two functions.

Let y be differentiable function of x such that y = u x v

Then we write

u_1, u_2, u_3 ..... u_n as first, second, third, ... nth order derivative of u.

v_1, v_2, v_3 - v_n as first, second, third, ... nth order derivative of v.

y_1, y_2, y_3 ..... y_n as first, second, third, ... nth order derivative of y.

Structure A : For first order derivative we write [Ref. Ex 1].

Coefficients	$1c_0 = 1$	$1c_1 = 1$
	u	u_1
	v	v_1

Hence, $y_1 = (1)\, u v_1 + (1)\, u_1 v = u v_1 + u_1 v.$

Structure B : For second order derivative we write [Ref. Ex 2].

Coefficients $\quad {}^2c_0 = 1 \quad {}^2c_1 = 2 \quad {}^2c_2 = 1$

$$\begin{array}{ccc} u & u_1 & u_2 \\ v & v_1 & v_2 \end{array}$$

$$y_2 = (1)\,u\,v_2 + (2)\,u_1 v_1 + (1)\,u_2 v$$

$$y_2 = u\,v_2 + 2\,u_1 v_1 + u_2 v.$$

Structure C : For third order derivative we write [Ref. Ex 3].

Coefficients $\quad {}^3c_0 = 1 \quad {}^3c_1 = 3 \quad {}^3c_2 = 3 \quad {}^3c_3 = 1$

$$\begin{array}{cccc} u & u_1 & u_2 & u_3 \\ v & v_1 & v_2 & v_3 \end{array}$$

$$y_3 = (1)\,u\,v_3 + (3)\,u_1 v_2 + (3)\,u_2 v_1 + (1)\,u_3 v$$

$$y_3 = u\,v_3 + 3\,u_1 v_2 + 3\,u_2 v_1 + u_3 v$$

Structure D : For n th order derivative we write coefficients, functions and their successive derivatives as :

$$\begin{array}{cccccccc} {}^nc_0 & {}^nc_1 & {}^nc_2 & {}^nc_3 & {}^nc_4 & \text{———} & {}^nc_n \\ u & u_1 & u_2 & u_3 & u_4 & \text{———} & u_n \\ v & v_1 & v_2 & v_3 & v_4 & \text{———} & v_n \end{array}$$

Hence, $y_n = {}^nc_0\,u\,v_n + {}^nc_1\,u_1 v_{n-1} + {}^nc_2\,u_2 v_{n-2} + {}^nc_3\,u_3 v_{n-3} + \text{—} + {}^nc_n\,u_n v.$

Remark. This is the famous Leibnitz's theorem for finding the n^{th} order derivative of product of two functions.

Illustrative Examples :

Ex 5. If $y = (x+1)^2 \cos x$, Find y_1 and y_2.

Ans : We write structure A and B together as follows

$u = (x+1)^2 \qquad u_1 = 2(x+1) \qquad u_2 = 2$

$v = \cos x \qquad v_1 = -\sin x \qquad v_2 = -\cos x$

Hence from structure A we get

$$y_1 = (1)(x+1)^2(-\sin x) + (1)\ 2(x+1)\cos x$$
$$= -(x+1)^2 \sin x + 2(x+1)\cos x.$$

And from structure B we get

$$y_2 = (1)[(x+1)^2(-\cos x)] + (2)[2(x+1)(-\sin x)] + (1)[2\cos x]$$
$$= -(x+1)^2 \cos x - 4(x+1)\sin x + 2\cos x.$$

Ex 6. If $y = x^2 \log(x^3)$ show that $x\ y_3 = 6$.

Ans : We write structure C as follows.

$u = x^2 \qquad u_1 = 2x \qquad u_2 = 2 \qquad u_3 = 0$

$v = \log(x^3) \quad v_1 = 3/x \qquad v_2 = -3/x^2 \qquad v_3 = 6/x^3$

$= 3\log x$

$$y_3 = (1)[x^2(6/x^3)] + (3)[2x(-3/x^2)] + (3)[2(3/x)] + (1)[0\ 3\log x]$$
$$= 6/x - 18/x + 18/x + 0 = 6/x. \quad \text{Thus} \quad x\,y_3 = 6$$

<u>Remark</u> : This structure helps us in evaluating the third derivative of y directly without evaluating the first and second derivative.

Ex 7. $y = (x-1)^3(2x^2+3x+1)$. Find y_1, y_2 and y_3 at $x = 0$.

Ans : We write structure for y_3.

$u = (x-1)^3 \qquad u_1 = 3(x-1)^2 \qquad u_2 = 6(x-1)$

$u_3 = 6$

$= -1 \qquad = 3 \qquad = -6$

$= 6$

For $x = 0$

$v = 2x^2+3x+1$ $\quad v_1 = 4x+3$ $\quad v_2 = 4$

$v_3 = 0$

$= 1$ $\quad = 3$ $\quad = 4$

$= 0$

For $x = 0$

Now from structure A we get

$y_1 = (1)(-1)(3) + (1)(3)(1) = -3 + 3 = 0$

From structure B we get

$y_2 = (1)(-1)(4) + (2)(3)(3) + (1)(-6)(1) = -4+18-6 = 8$

From structure C we get

$y_3 = (1)(-1)(0) + (3)(3)(4) + (3)(-6)(3) + (1)(6)(1)$

$= 0 + 36 - 54 + 6 = -12.$

Thus $y_1 = 0, y_2 = 8, y_3 = -12$ for $x = 0$.

Ex 8. $y = x^3 e^{2x}$ Find y_n.

Ans : From structure D we get.

$u = x^3 \quad u_1 = 3x^2 \quad u_2 = 6x \quad u_3 = 6 \quad u_4 = 0 \ldots\ldots\ldots\ldots u_n = 0$

$v = e^{2x} \quad v_1 = 2e^{2x} \quad v_2 = 4e^{2x} \ldots\ldots\ldots\ldots v_{n-1} = 2^{n-1}e^{2x}, v_n = 2^n e^{2x}$

$y_n = {}^nc_0 (x^3)(2^n e^{2x}) + {}^nc_1 (3x^2)(2^{n-1} e^{2x}) + {}^nc_2 (6x)(2^{n-2} e^{2x}) + ({}^nc_3 .6 . 2^{n-3} . e^{2x}.)$

$= [x^3 2^n + 3n x^2 2^{n-1} + 3n(n-1) x 2^{n-2} + n(n-1)(n-2) 2^{n-3}] e^{2x}.$

Exercise 19.1

1) Find y_1 and y_2 if $y = e^{3x} \sin 4x$.

2) Find y_4 if $y = (\log x)^2 x^3$.

3) If $y = (2x + 1)^2 (1 - 3x)$ Find y_1, y_2, y_3 at $x = 1$.

4) If $y = x^5 \sin x$ Find y_1 and y_2.

5) If $y = (x + 1)(x - 2)^4$ Find y_1 and y_3 at $x = 0$.

6) $y = e^{-x} \cos x$ Show that $y_4 + 4y = 0$.

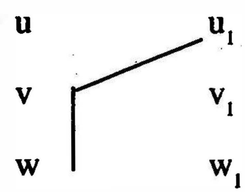
7) Differentiate n times the equation.

$(1 - x^2)\, y_2 - x\, y_1 + a^2 y = 0.$

19.4 If y is differentiable function of x such that,

$y = u\ v\ w$ then we write the structure [Ref. Ex 4], as.

Structure E :

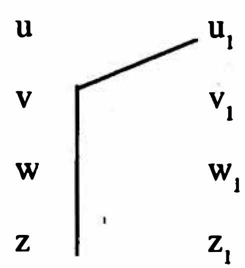


and apply the part II procedure of Ex 4. We get

$$y_1 = \sum u_1 v\, w = u_1 v\, w + v_1 u\, w + w_1 u\, v$$

Similarly for $y = u\ v\ w\ z$ we write the structure as.

Structure F **:**



Hence $y_1 = u_1 v\, w\, z + v_1 u\, w\, z + w_1 u\, v\, z + z_1 u\, v\, w$.

Note **:** This method can be further extended to find first derivative when y is product of n functions. Note that u_n is n^{th} derivative of u with respect to x

Illustrative Examples

Ex 9. $y = (3x + 1)(x + 2)^2 (x^2 - 2)$, Find y_1.

Ans : Ref. structure E. We write.

$$u = 3x + 1 \qquad u_1 = 3$$

$$v = (x + 2)^2 \qquad v_1 = 2(x + 2)$$

$$w = x^2 - 2 \qquad w_1 = 2x$$

Thus $y_1 = 3(x + 2)^2 (x^2 - 2) + 2(x + 2)(3x + 1)(x^2 - 2) + 2x(3x + 1)(x + 2)^2$.

Ex 10. $y = 3x^2 (2x - 1)^2 (1 - 4x)$ Find y_1 at $x = 1$

Ans : We write

$$u = 3x^2 \qquad u_1 = 6x$$

$$= 3 \qquad = 6 \text{ For } x = 1$$

$$v = (2x - 1)^2 \qquad v_1 = 4(2x - 1)$$

$$= 1 \qquad = 4 \text{ For } x = 1$$

$$w = (1 - 4x) \qquad w_1 = - 4$$

$$= - 3 \qquad = - 4 \text{ For } x = 1.$$

Thus $y_1 = 6 \ 1 \ (- 3) + 4 \ (3) \ (- 3) + (- 4) \ 3 \ (1)$

$= - 18 - 36 - 12$

$y_1 = - 66.$

Ex 11. $y = e^{-x} (x^2 \sin x + \cos 2x)$

Ans : We combine the structure E and A to write.

$$u = e^{-x} \qquad u_1 = -e^{-x}$$

$$v = x^2 \qquad v_1 = 2x$$

$$w = \sin x \qquad w_1 = \cos x$$

$$u = e^{-x} \qquad u_1 = - e^{-x}$$

$$z = \cos 2x \qquad z_1 = -2 \sin 2x.$$

Hence $y_1 = (-e^{-x})\,(x^2 \sin x) + (2x\ e^{-x}\ \sin x) + \cos x\ e^{-x}\ x^2$

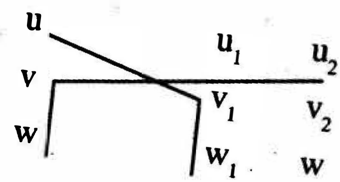
$+ e^{-x}(-2 \sin 2x) + \cos$
$2x\ (-e^{-x})$

$= e^{-x}(-x^2 \sin x + 2x \sin x + x^2 \cos x - 2\sin 2x - \cos 2x).$

19.5 When y is differentiable function of x such that $y = u$
$v\ w$,

we write the structure as.

Structure G :



Thus $y_2 = u_2 v w + 2\ u v_1 w_1$

or $y_2 = u_2 v w + v_2 u w + w_2 u v + 2\,(u v_1 w_1 + v u_1 w_1 + w u_1 v_1)$

Illustrative Examples

Ex 12. $y = e^{x/2} . x^2 . \cos x$, Find y_2.

Ans : We write the structure as.

$u = e^{x/2}$	$u_1 = 1/2\ e^{x/2}$	$u_2 = 1/4\ e^{x/2}$
$v = x^2$	$v_1 = 2x$	$v_2 = 2$
$w = \cos x$	$w_1 = \sin x$	$w_2 = -\cos x$

Thus $y_2 = (1/4)\, e^{x/2} . x^2 \cos x + 2 . e^{x/2} . \cos x + (-\cos x) . x^2 e^{x/2}$

$+ 2\,[e^{x/2} . 2x\,(-\sin x) + x^2\,(1/2)\, e^{x/2} . (-\sin x) + \cos x\,(1/2)\, e^{x/2} . 2x]$

$y_2 = e^{x/2}\,[\,(1/4)\, x^2 .\cos x + 2\cos x - x^2 .\cos x - 4x \sin x - (1/2)\, x^2 .\sin x + x \cos x\,].$

Ex 13. $y = (x^2 + 2x - 1)(3x + 1)(x^2 + 2)$. Find y_1 and y_2 at $x = 0$.

$u = x^2 + 2x - 1$	$u_1 = 2x + 2$	$u_2 = 2$
$= -1$	$= 2$	$= 2$ For $x = 0$.
$v = 3x + 3$	$v_1 = 3$	$v_2 = 0$
$= 1$	$= 3$	$= 0$ For $x = 0$.
$w = x^2 + 2$	$w_1 = 2x$	$w_2 = 2$
$= 2$	$= 0$	$= 2$

For $x = 0$.

Hence

1) $y_1 = 2 \,.\, 1 \,.\, 2 + 3 \,.\, (-1)\ 2 + 0 = 4 - 6 = -2$. at $x = 0$.

2) $y_2 = 212 + 0 + 2(-1)2 + 2\ [\ (-1)30 + 120 + 223\ ]$

$= -4 + 24 = 20$ at $x = 0$.

Remarks :-

(1) Example 5 to 13 reflects the effectiveness of the structural method over the current methods.

(2) This structural method is most useful when derivatives are evaluated at particular values of x. (Ref. 7 , 10 , 13).

(3) Readers are requested to solve these examples by current methods and compare with structural methods.

Exercise **19.2**

1) Find y_1 and y_2 if $y = (ax + b)(cx + d)(ex + f)$ where a, b, c, d, e, f are constants.

2) Find y_2 if $y = 7^x \,.\, x^7 \,.\, \log 7x$.

3) Find y_1 and y_2 if $y = (x^2 + 1)(3x - 2)(5 - x)$ at $x = 1$.

4) Find y_2 if $y = e^{-x} \,.\, \log x \,.\, \tan x$.

5) $y = x^3 [(Dx + E)(x^2 + 1)^2 + (Fx + G)(x + 1)]$,
Find y_1.

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CHAPTER 20.

INTEGRATION

20.1 Integration is the converse process of differentiation. The standard rules of integration can be interpreted by means of VM. The basic concept of integration is given by Reimann in which the integration is a limiting sum (upper and lower). In this chapter we shall study the methods of integrals of products of two functions by following sutra.

Sutra : ऊर्ध्वतिर्यग्भ्याम् Urdhva tiryakbhyam

(Vertically and crosswise)

This method is much simpler than the methods discussed in textbooks. Here it is assumed that readers are familiar with integrals of standard functions and Rules and Methods of integration.

Method :- To find $\int (u \quad v)\, dx$, we write the structure as follows.

1) The successive derivatives of u are written in first row.

2) The successive integrals of v are written in second row.

3) Apply Tiryak sutra with $(-1)^{n-1}$ where n is the order of Tiryak.

4) Find integral of Urdhva product of last term of structure.

Thus, if $u \quad u_1 \quad u_2 \quad u_3 \quad u_4 \; ..... \; u_n$

are successive derivatives of u and,

$$v \quad v \quad v \quad v \quad v \cdots v^n$$

are successive integrals of v, then we write.

$$u \quad u_1 \quad u_2 \quad u_3 \quad u_4 \ldots\ldots u_n$$
$$v \quad v \quad v \quad v \quad v \ldots\ldots v^n$$

And $\int (u \times v)\, d\, x = (-1)^0 u\, v + (-1)^1 u_1\, v + (-1)^2 u_2\, v + (-1)^3 u_3\, v + \text{———} + (-1)^n\ u_n\, v^n dx.$

Illustrative Examples

Ex (1) Find $I = \int x^4 . e^x dx.$

Ans : We write the structure as follows.

$$u = x^4 \quad u_1 = 4x^3 \quad u_2 = 12x^2 \quad u_3 = 24x \quad u_4 = 124$$
$$v = e^x \quad v = e^x \quad v = e^x \quad v = e^x \quad v = e^x$$

Hence, $I = (-1)^0 x^4 e^x + (-1)^1\, 4x^3 e^x + (-1)^2\, 12x^2 e^x + (-1)^3\, 24x\, e^x$

$+ (-1)^4\ 24\, e^x dx$

$= e^x [\, x^4 - 4x^3 + 12x^2 - 24x + 24\,] + c.$

Ex (2) Find $I = \int x^2\ \cos 3x\, dx$

Ans : We write the structure as,

$$u = x^2 \qquad u_1 = 2x \qquad u_2 = 2$$
$$v = \cos 3x \qquad v = (1/3)\sin 3x \qquad v = (-1/9)\cos 3x$$

$I = (-1)^0 x^2 (1/3) \sin 3x + (-1)^1 2x\ (-1/9) \cos 3x$

$+ (-1)^2\ (-1/9) \cos 3x\ 2dx$

$I = (1/3)\, x^2 \sin 3x + (2/9)\, x\ \cos 3x - (2/27) \sin 3x + c$

Ex (3) Finh $I = \int e^{ax}.\cos bx\, dx$ — (Reduction Formula)

Ans : We write the structure as,

$u = \cos bx$ $u_1 = - b \sin bx$ $u_2 = - b^2 \cos bx$

$v = e^{ax}$ $v = (1/a)\, e^{ax}$ $v = (1/a^2)\, e^{ax} I = (-1)^0$
$(1/a) \cos bx\, e^{ax} + (-1)^1 (- b/a^2) \sin bx\ e^{ax}$

$+ (-1)^2 (- b^2/a^2) [\ e^{ax} \cos bx\, dx\ ]$

$I = e^{ax} [\ (1/a) \cos bx + (b/a^2) \sin bx] - (b^2/a^2) [\ I\ ]$

Hence, $(a^2 + b^2)\, I = e^{ax} [\ a \cos bx + b \sin bx\ ]$

$I = e^{ax} [\ a \cos bx + b \sin bx\ ] / (a^2+b^2)$

20.2 Integrals by partial fractions

Sutra : परावर्त्य योजयेत् Paravartya yojayet

(Transpose and apply)

Ex (4) Find $I = \int x+3 / (x^2-3x+2)\, dx$

Ans : We write the integrand as,

$$\frac{x+3}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

Hence $x + 3 = A (x+ 2) + B (x - 1)$.

Put $x = 1$: $4 = A(3)$ Hence $A = 4/3$.

Put $x = 2$: $1 = B(- 3)$ Hence $B= 1/3$.

Thus $I = (4/3)\ [1/ (x - 1)]\, dx - (1 / 3)\ [\ 1 / (x+2)\ ]\, dx$

$$= (4/3) \log (x + 1) - (7/3) \log (x + 2) + c.$$

Exercise :-

Evaluate the following integrals.

1. $(x + 1) / (x^3 + x^2 + 6x)$

2. $x^3 / [(x-1)(x-2)(x-4)]$,

3. $(x^2 + 2x + 2) / [(x+1)(x+2)]$

4. $(x^3 + 2x^2 + 2x + 1) / [x(x+1)(x+2)(x+4)]$

Exercise :

Integrate the following w.r.to x using VM.

$x^4 e^x$, $x e^x$, $x^3 \cos x$, $x^4 \sin x$

$x \log x$, $x^2 \log x$, $e^{3x} \cos 4x$, $e^{3x} \sin 4x$.

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PART (V)

MISCELLANEOUS

CHAPTER 21.

COMPLEX NUMBERS

21.1 In this chapter we will study the elementary algebra of complex numbers with the help of concept of triplet associated with complex number.

21.2 Definition :- The number $\sqrt{-1}$ is called imaginary number, which is denoted by 'i'. Thus $i = \sqrt{-1}$ and $i^2 = -1$

Definition :- The number z of the type $z = a + b\,i$, where a and b are real numbers and i is imaginary number, is called complex number. Here a is called real part (R.P.)and b is called imaginary part (I.P.) of complex number z

e.g. :-

complex number	R.P.	I.P.
$3 + 4i$	3	4
$2 - 7i$	2	-7
$6i$	0	6
8	8	0

Note :

[1] $8 = 8 + 0\,i$. Hence every real number is a complex number with imaginary part 0. Thus set of real numbers is sub set of set of complex numbers.

[2] When R.P. is 4 and I.P. is (- 3) then the corresponding complex number is $4 - 3i$

[3] Zero is a complex number as : $0 = 0 + 0i$

21.3 Equality of complex numbers.

Two complex numbers are said to equal if their corresponding real and imaginary parts are equal.

e.g. Let $z_1 = a + bi$, $z_2 = c + di$ are any two complex numbers

Then $z_1 = z_2$ if and only if $a = c$, and $b = d$.

If $x + yi = 2 - 7i$ then $x = 2$ and $y = -7$.

21.4 Modulus of a complex number.

The modulus, $|z|$, of any complex number $z = a + bi$ is defined as

$$|z| = \sqrt{(a^2 + b^2)}.$$

Note :- $|z|$ is always positive.

For $z = 2 - 3i$, $|z| = \sqrt{[2^2 + (-3)^2]} = \sqrt{(13)}$.

21.5 Conjugate of a complex number.

The conjugatez of any complex number $z = a + bi$ is defined as $\bar{z} = a - bi$.

complex number	conjugate
$2 + 3i$	$2 - 3i$
$-3 + i$	$-3 - i$
$-5 - 7i$	$-5 + 7i$
$2i$	$-2i$

etc.

Note:- [1] we have $|z| = |\bar{z}| = (\sqrt{a^2 + b^2})$.

21.6 Triplets corresponding to complex number.

We have already studied triplets corresponding to angles in the triangle such as,

T (A) : $[x_1, y_1, r_1]$ T (B) : $[x_2, y_2, r_2]$

Where $r_1^2 = x_1^2 + y_1^2$, and $r_2^2 = x_2^2 + y_2^2$, and also

$$T(A + B) = [\ x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1, r_1 r_2\ ]$$

$$T(A - B) = [\ x_1 x_2 + y_1 y_2, x_2 y_1 - x_1 y_2, r_1 r_2\ ]$$

Now we define triplet corresponding to complex number.

***Def*:—** The triplet T (z) corresponding to complex number $z = x + y\,i$ is defined as $T(z) : [\,x, y, r\,]$

where x is real part, y is imaginary part of the complex number z and $r = |z| = \sqrt{(x^2 + y^2)}$

For example:

[1] For $z = 3 + 4\,i$, $T(z) : [\,3, 4, 5\,]$

[2] For $z = 1 - 3\,i$, $T(z) : [\,1, -3, 10\,]$

[3] For $z = 8\,i$ $T(z) : [\,0, 8, 8\,]$

[4] For $T(z) = [-3\ \ 2\ \ \sqrt{13}]$, $z = -3 + 2\,i$.

[5] For $z_1 = z_2$, $T(z_1) = T(z_2)$.

21.7 Addition and Subtraction of Complex Numbers.

Definition:— The addition and subtraction of two complex numbers $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ is defined as

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2) i$$

$$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2) i.$$

$$\therefore T(z_1 \pm z_2) = [x_1 \pm x_2, \ y_1 \pm y_2 \ r]$$

Thus we add corresponding real parts and imaginary parts.

Note that for angles A and B, the rules for T (A ± B) are different from those of $T(z_1 \pm z_2)$.

Ex. Find $z_1 + z_2$ if $z_1 = 7 + 4i$, $z_2 = -2 + 6i$

Ans. We write,

$T(z_1)$:	[7	4	-]
$T(z_2)$:	[-2	6	-]
$T(z_1 + z_2)$:	[5	10	-]
$T(z_1 - z_2)$:	[9	-2	-]

Hence $\mathbf{z_1 + z_2 = 5 + 10\,i}$, and $\mathbf{z_1 - z_2 = 9 - 2\,i}$

Note:-

[1] We write the third element of triplet, if necessary..

[2] Addition of complex numbers is commutative, and associative i.e.

$z_1 + z_2 = z_2 + z_1$ and $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

[3] For any complex number z, $z + 0 = 0 + z = z$.

[4] $z_1 + z_2 = z_2 + z_1$.

[5] For $z = x + y i$, $z + \overline{z} = 2x$. and $z - \overline{z} = 2 y i$

21.8 Multiplication of complex numbers

The method of multiplication of two complex numbers is exactly similar to addition of triples of angles.

To find $z_1 . z_2$ If $z_1 = x_1 + y_1 i$, and $z_2 = x_2 + y_2 i$

We write,

$$T(z_1) \quad : [x_1 \quad y_1 \quad r_1]$$

$$T(z_2) \quad : [x_2 \quad y_2 \quad r_2]$$

Hence $T(z_1 z_2) : [x_1x_2 - y_1y_2 \qquad x_2y_1 + x_1y_2 \qquad r_1r_2]$

Thus $z_1 z_2 = (x_1 x_2 - y_1 y_{12}) + (x_2 y_2 + x_1 y_2) i$.

Note:-

[1] In numerical problems we need not find r_1, r_1 and r_2 r_2.

[2] Readers will understand that this is an extension of familiar

urdhva-tiryak sutra.

[3] $|z_1||z_2| = |z_1 z_2|$.

21.9 Properties of a product of complex numbers.

[1] The multiplication of complex numbers is commutative and associative.(The proof is left to the readers.)

[2] The multiplication is distributive over addition.

Proof :- Let $z_1 = x_1 + i y_1$, $z_2 = x_2 + i y_2$, $z_3 = x_3 + i y_3$

are any three complex numbers.

To show that $z_1 (z_2 + z_3) = (z_1 z_2) + (z_1 z_3)$

Let $T(z_1)$: $[x_1 \quad y_1 \quad -]$

$T(z_1)$: $[x_2 \quad y_2 \quad -]$

and $T(z_3)$: $[x_3 \quad y_3 \quad -]$

Then $T(z_1+z_3)$: $[x_1 + x_3 \quad y_1+y_3 \ -]$
$= [a \quad b \quad -]$

Where $a = (x_1+x_3)$, and $b = y_1+y_3$.

and $T\{z_1 (z_2+z_3)\}$: $[x_1 a - y_1 b \quad a y_1+x_1 b -]$

$= [c \quad d \quad -]$

Thus $z_1 (z_2 + z_3) = c + d i$ (1)

Further $T(z_1 z_2)$ $[x_1 x_2 - y_1 y_2 \quad x_2 y_1 + x_1 y_2 -]$

and $T(z_1 z_3)$: $[x_1 x_3 - y_1 y_3 \quad x_3 y_1 + x_1 y_3 -]$

Then $T\{(z_1 z_2) + (z_1 z_3)\}$

: $[x_1(x_2+x_3)-y_1(y_2+y_3) \quad y_1(x_2+x_3)+x_1(y_2+y_3) -]$

: $[x_1 a - y_1 b \quad y_1 a + x_1 b \ -] = [c \quad d \quad -]$

Thus $(z_1 z_2) + (z_1 z_3) =$ $c + d i$ (2)
From (1) and (2) we get $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Similarly we can show that $z_1 (z_2 - z_3) = z_1 z_2 - z_1 z_3$

[3] Show that for any complex number z,

$z.1 = z$ and $z.0 = 0$.

Note: $1 = 1 + 0\,i$ and $0 = 0 + 0\,i$

[4] Show that for any complex number z, $z\bar{z} = |z|^2$.

Let $z = x + y\,i$ and $\bar{z} = x - y\,i$ then we write

$T(z)$	: [x	y	r]
$T(\overline{z})$	: [x	$-y$	r]
Then $T(\overline{zz})$	: [$x^2 + y^2$	$xy - xy$	r^2]

Hence $z.\overline{z} = x^2 + y^2 = |z|^2$.

Illustrative Examples.

[1] Find (3 + 4 i) . (1 + 7 i).

let $z_1 = 3 + 4i$ and $z_2 = 1 + 7i$,

Then $T(z_1)$:	[3	4	-]
$T(z_2)$:	[1	7	-]
Hence $T(z_1 . z_2)$:	[3-28	4-21	-]
	[-25	-17	-]

Thus **$(z_1 z_2) = -25 - 17i$**

[2] Find (1 + i) X (1 - 3 i) X (-2 + 5 i).

We write $z_1 = 1 + i$, $z_2 = 1 - 3i$

$T(z_1)$:	[1	1	-]
$T(z_2)$:	[1	- 3	-]

Hence

$T(z_1 z_2)$:	[4	- 2	-]

$T(z_3)$: [- 2   5   -]

$T(z_1 z_2 z_3)$: [2   24   -]

Thus $(z_1 z_2 z_3) = 2 + 24\,i$.

21.10 Square of the complex number.

To find z^2 if $z = x + y\,i$.

We write $T(z)$: [x   y   r]

$T(z)$: [x   y   r]

[$x^2 - y^2$, $2xy$, r^2]

Thus $z^2 = (x^2 - y^2) + (2xy)\,i$.

e.g. For $z = 3 + 4i$, $z^2 = (3^2 - 4^2) + (2 . 3 . 4)\,i = -7 + 24$
i

Ex. [3] Show that $w = 1/2 \{ -1 +\ \ 3\,i \}$ is cube root of unity.

Here we show that $w^3 = 1$.

We have $T(w)$: [-1/2   √ 3/2 -]

$T(w^2)$: [- 1/2   - 3/2   -]

Hence $T(w^3)$: [1/4 - (- 3/4)-3/2 + 3/2   -]

[= 1   = 0   -]

Thus $w^3 = 1 + 0\,i = 1$.

We observe that $w^2 = (-1/2) - (3/2)\,i = w$

21.11 Additional cases .

Ex **[4]** Find (-2 + 3 i) (3 - 2 i) + (1+ i) (4 + i) - (3 - 2 i) (2 + 5 i)

We write

$T(z_1)$	:	[2	3	-]
$T(z_2)$	:	[3	-2	-]
$T(z_1 z_2)$	:	[12	5	-]
$T(z_3)$	:	[1	1	-]
$T(z_4)$	:	[4	1	-]
$T(z_3 z_4)$	:	[3	5	-]
$T(z_5)$	:	[3	-2	-]
$T(z_6)$	:	[2	5	-]
$T(z_5 z_6)$	:	[16	11	-]

Adding and subtracting (12 + 3 - 16) (5 + 5 - 11)

= -1 -1

Hence answer is **(- 1 - i).**

Ex. [5] If x = 2 + 3 i, show that

$x^2 - 5x + 15 = -3i$ and

$x^3 - 4x^2 + 13x = 0$.

We write

$T(x)$	:[2	3	-]
$T(x^2)$	:[-5	12	-]

	$T(-5x)$	:[-10	-15	-]
	$T(15)$	:[15	0	-]

Adding we get, $T(x^2 - 5x + 15)$:[0   -3   -]

Hence $x^2 - 5x + 15 = 0 - 3i = -3i$.

Further	$T(x^3)$	: [- 46	9	-]
	$T(-4x^2)$	: [20	- 48	-]
	$T(13x)$	: [26	39	-]
Hence	$T(x^3 - 4x^2 + 13x)$	: [0	0	-]

Hence $x^3 - 4x^2 + 13x = 0$.

21.12 Division of complex numbers.

To find z_1 / z_2 if $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

We write $z_1 / z_2 = (z_1 z_2) / (z_2 z_2)$

and	$T(z_1)$	:	$[x_1, \; y_1 -]$
	$T(z_2)$	:	$[x_2 \quad y_2 \quad -]$
	$T(z_2)$	:	$[x_2 \quad -y_2 \quad -]$
	$T(z_1 z_2)$	:	$[x_1 x_2 - y_1 y_2 + x_2 y_1 + x_1 y_2 -]$
	$T(z_2 z_2)$	:	$[x_2^2 + y_2^2 \quad 0 \quad -]$

Hence $z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_2 y_1 + x_1 y_2) i$

and $z_2 z_2 = (x_2^2 - y_2^2) + 0i$

Thus $z_1 / z_2 = \{ (x_1 x_2 - y_1 y_2) + (x_2 y_1 + x_1 y_2) i \} / (x_2^2 + y_2^2)$

Readers will note that the value of numerator of $T(z_1 / z_2)$ is

exactly similar to T (A - B) for angles A and B.

Ex. [6] Find (1 + i) / (3 - i).

We write	$T(z_1)$: [1		-1-]	
	$T(z_2)$: [3		-1	-]
	$T(z_2)$: [3		1	-]
	$T(z_1 z_2)$	: [2		4 -]
	$T(z_2 z_2)$	: [10	0 -]	

Hence : $z_1 / z_2 = (2 + 4\,i) / 10 = \mathbf{(1/5) + (2/5)\,i}$

Ex.[7] Express in the form of a + i b.

{ (1 + 3 i)(2 + 6 i) / (3 -2i) }

$T(z_1)$	:	[1	3	-]
$T(z_2)$	:	[2	6	-]
$T(z_1 z_2)$	:	[-16	12	-]

$T(z_3)$:	[3	-2	-]
$T(z_3)$:	[3	2	-]

$T(z_1 z_2 z_3)$:	[-48-24	-32 +36	
$T(z_1 z_2 z_3)$:	[= -72= 4	-]	

$T(z_3 z_3)$	:	[13	0	-]

Hence $(z_1 z_2) / z_3 = \mathbf{(-72/13) + (4/13)\,i.}$

21.13 To find the reciprocal of z = x + y i, z 0

We write	1 / z = (1 + 0 i) / (x + y i) .				
Now	T (1 + 0 i)	:	[1	0	-]
	T (z)	:	[x	y	-]
	T (z)	:	[x	- y	-]

$$T(1 X z) : [x \quad -y \quad -]$$
$$T(z\ z) : [(x^2+y^2) \quad 0-]$$

Hence $\mathbf{1/z = (x - y i)/ x^2 + y^2}$

Note : $1/z = z / |z|^2$.

21.14 To find the SQUARE ROOT of the complex number $z = x + y i$

Let $\sqrt{x + y i} = (a + b i)$, Where a and b are real numbers. Squaring we get, $(x + y i)^2 = a^2 - b^2 + 2 a b i$

Equating real and imaginary parts we get, $x = a^2 - b^2$ and $y = 2 a b$.

For $b = y / (2 a)$ we get $x = a^2 - (y^2 / 4 a^2)$

Simplifying we get, $4 a^4 - 4 x a^2 - y^2 = 0$.

We consider this as quadratic equation in a^2 and hence roots are

$$a^2 = (1/8) \{ 4 x \pm \sqrt{(16 x^2 + 16 y^2)} \}$$

Hence $a^2 = (1/2) \{ x \pm \sqrt{(x^2 + y^2)} \}$

Now for $T(z) : [x \quad y \quad r]$

where $r = (x^2 + y^2)^{1/2}$ gives $a^2 = (x \quad r) / 2$

Thus $a = \sqrt{(x + r)/2}$ and $a = \sqrt{(x - r)/2}$

For triplet $[x, y, r]$ we have $x < r$ for all x and r

Hence $x - r < 0$ for all x and r.

As a is real number we delete the value $a = \sqrt{(x - r)/2}$

Thus only value of a is $a = \sqrt{(x + r)/2}$

Then $b = y / (2a)$.

Hence $(x^2 + y^2) = a + b i$ with above values of a and b and where r is third element of T (z) .

Ex.[8] :- Find the square root of $-24 + 70i$

Let $x + y i = -24 + 70 i$ and $\sqrt{x + y i} = \pm (a + b i)$

Now $T(-24 + 70 i) : [-24 \quad 70 \quad r]$

Now $r^2 = (-24)^2 + (70)^2 = 5476$ Hence $r = 74$.

Thus $T(-24 + 70 i) = [-24 \quad 70 \quad 74]$

and $x + r = 50$.

Hence $a = \sqrt{(x + r)/2} = \sqrt{(50/2)} = 5$, and

$b = y / 2a = 70 / 10 = 7$.

Hence roots are $(5 + 7 i)$.

EXERCISE

Set A Multiply

[1] $(1+ 3 i)(2 + 5 i)$ [2] $(2 - i)(2 + 6 i)$

[3] $(2+4 i)(3 + 5 i)$. [4] $i(1 - i)(1 + 2 i)(2+i)$

[5] $(2-4i)(1+6i)(7-3i)(2i)$.

Set B Find the value of :—

[1] $(11+2i)(3+i)-(4+i)(8-i)$

[2] $(3+i)(4-i)+(7-i)(4+3i)$

[3] $(1+i)(2-i)-(1+3i)-(2+7i)(4+3i)$

[4] $(2+i)^2$ [5] $(1-3i)^3$

Set C

[1] If $w = \{-1+(3)^{1/2} i\}/2$, show that

$w^2 = w$ and $w^2 = w$. Also show that $w^3 = 1$.

[2] If $x = 2+3i$, Show that $x^2 - 4x + 13 = 0$.

[3] If $x = 1-(3)^{1/2} i$ Show that $x^3 - x^2 + 2x + 4 = 0$.

[4] If $x = 3+2i$, Show that $x^4 - 3x^3 - 2x^2 + 21x + 39 = 0$.

[5] If $x = 3-i$. Show that $x^3 - 3x^2 - 9x + 33 = i$

Set D Express in the form of $a + bi$.

[1] $(4+2i)/(3+2i)$ [2] $(6-i)/(2+5i)$.

[3] $(2+i/2-i)^2$ [4] $(1+2i)(2-3i)/(3+4i)$.

[5] $1/(7+4i)$ [6] $(1+5i)/\{(1-i)(2+3i)\}$.

<u>*Set E*</u> ***Find the square root of : —***

[1] $12 + 5i$ [2] $(33 + 56i)/4$

[3] $-7 + 24i$ [4] $15 - 8I$

[5] $-24 + 70i$. [6] $-7 + 6\sqrt{2}\, i$

ANSWERS

<u>*Set A*</u>

[1] $-13+11i$ [2] $10 + 10i$ [3] $-14 +22i$

[4] $-5 + 5i$ [5] $44 + 412i$

Set B

[1] $-2 + 13i$ [2] $44 + 18i$ [3] $7 - 14i$

[4] $3 + 4i$ [5] $-26 + 18i$

<u>*Set D*</u>

[1] $(16/13) - (2/13)i$ [2] $(7/29) - (32/29)i$

[3] $(-7 + 12i)/625$. [4] $(28 - 29i)/25$

[5] $(7 - 4i)/65$ [6] $(5 + 12i)/13$.

<u>*Set E*</u>

[1] $\pm(5 + i)/$ I 2 [2] $\pm(7/2) + 2i$

[3] $\pm (3 + 4i)$ [4] $4 - i$.

[5] $\pm (5 + 7i)$ [6] $\sqrt{2} + 3i$.

❖ ❖ ❖

CHAPTER 22.

DETERMINANTS

22.1 In this chapter we shall study the evaluation of determinants of second, third and forth order by URDHVA TIRYAK (vertically & crosswise) formula. We shall also study the different applications of determinants in solving simultaneous equations, products of vectors, pair of lines etc. It is assumed that readers are familiar with basics of determinants and its applications.

SUTRA ऊर्ध्वतिर्यग्भ्याम् Urdhva tiryakbhyam

(Vertically and crosswise)

22.2 Evaluation of Second Order Determinant

We know, $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1 a_2$

Here, Urdhva Tiryak (crosswise) formula can be used.

Ex. 1. Evaluate $\begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix}$

Here, $D = (3 . 5) - (-2 . 4) = 23.$

Ex. 2. Evaluate $\begin{vmatrix} x+1 & 2x+3 \\ x-2 & 3x+2. \end{vmatrix}$

$$\begin{array}{lccc} \text{Here } D = & x+1 & & 2x+3 \\ \times & 3x+2 & + & x-2 \\ \hline & & 5x^2+4x-4 & \end{array}$$

22.3 Evaluation of Third Order Determinant

To find the value of the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

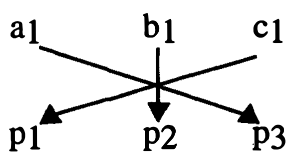
Row Expansion Method :-

Step 1 : - We write elements of first row as base and three second order determinants from elements of second and third row and evaluate them.

$$\begin{array}{ccc} a_1 & b_1 & c_1 \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \\ p_1 & p_2 & p_3 \end{array}$$

(where p_1, p_2, p_3 are the values of second order determinants)

Step 2 :- Apply Sutra { 1 } to structure



then $D = a_1 p_3 - b_1 p_2 + c_1 p_1$.

Remark :- Determinant can be evaluated if second or third row is considered as a base

22.4 Column Expansion Method

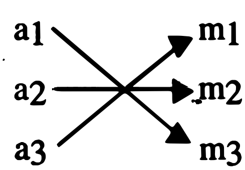
Step 1:- we write the elements of the first column as base and three-second order determinants from elements of second and third column and evaluate them.

$$a_1 \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = m_1$$

$$a_2 \quad \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = m_2$$

$$a_3 \quad \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = m_3$$

Step 2 :- We apply Tiryak sutra for the structure :



Then $D = a_1 m_3 - a_2 m_2 + a_3 m_1$.

Remarks -

[1] Similar method follows if second or third column is considered as a base.

[2] We get same value of the determinant irrespective of the method or the base.

Ex. 3: Find the value of $D = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 5 & -1 & 2 \end{vmatrix}$

A) Row expansion method.

Step 1:-

$$\begin{array}{ccc} 1 & 2 & 4 \\ \begin{vmatrix} 2 & 1 \\ 5 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} \end{array}$$

Step 2:- We get

$$\begin{array}{ccc} 1 & 2 & 4 \\ -7 & -11 & 5 \end{array}$$

Step 3:- Apply sutra {1} then $D = (1)(5) - (2)(-11) + (4)(-7)$
$= -1.$

Note :- The values of the second order determinants can be evaluated orally to write the step 2 directly.

B) Column Expansion Method

$$1 \quad \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2$$

$$2 \quad \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 8$$

$$5 \quad \begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} = 5$$

Thus $D = (1)(5) - (2)(8) + (5)(2) = -1$

Ex. 4 : Find the value of x if $D = \begin{vmatrix} 0 & -x & 2 \\ 1 & 0 & 3 \\ 4 & -1 & -1 \end{vmatrix} = 11$

Answer : By row expansion method we write,

$$\begin{matrix} 0 & x & 2 \\ -1 & -13 & — \end{matrix}$$

{ We need not evaluate third determinant because there is a zero at the corresponding cross-place }.

Thus $D = -2 - 13x$.

Hence $-2 - 13x = 11$ gives $x = -1$

Ex. 5 Evaluate $\begin{vmatrix} 2 & -1 & 4 \\ 7 & -3 & -2 \\ 4 & 1 & 8 \end{vmatrix}$

Answer : -

Apply $R_2 - 3R_1$ and $R_3 + R_1$ then

$$D = \begin{vmatrix} 2 & -1 & 4 \\ 1 & 0 & -14 \\ 6 & 0 & 12 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 2 & 1 & -4 \\ 1 & 0 & 14 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 6 \begin{array}{ccc} 2 & 1 & 4 \\ 0 & 16 & 0 \end{array}$$

$$= 6\ (16)$$

$$= 96$$

Note :

[1] Elementary row / column operations do not change the value of a determinant.

[2] Always try to get maximum zeros wherever possible by elementary row or column operations so that the calculation part becomes easier.

22.5 Properties of a Determinant

(1) The value of a determinant remains same when the rows and the columns are interchanged.

Proof :-

Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

By row expansion method , we get

$$D_1 = a_1p_3 - b_1p_2 + c_1p_1 \qquad \text{Where ,}$$

$$p_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad p_2 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \quad p_3 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

Interchanging rows and columns of D_1 we get

$$D_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

By column expansion method, we get

$$D_2 = a_1\, p_3 - b_1\, p_2 + c_1\, p_1$$

Thus $D_1 = D_2$

(2) The value of the determinant changes its sign when any two rows or columns are interchanged

(3) The value of a determinant changes its sign if any two rows or two columns are interchanged.

(4) If any row or column is multiplied by constant ' k ' then the value of a determinant becomes k times the original value.

{ The proofs are left as an exercise for the readers }.

(5) If any two rows or columns are identical then the value of the determinant is zero

{ The proof is left as an exercise }

22.6 Evaluation of Forth Order Determinant

Row Expansion Method

Let the determinant $D = \begin{vmatrix} a1 & b1 & c1 & d1 \\ a2 & b2 & c2 & d2 \\ a3 & b3 & c3 & d3 \\ a4 & b4 & c4 & d4 \end{vmatrix}$

Step 1 : We divide the determinant into two parts by a horizontal line.

$$D = \left|\begin{array}{cccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \hline a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{array}\right|$$

Step 2 : From elements of the upper part we construct six second order determinants and evaluate them

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = x_1, \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = x_2, \quad \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} = x_3$$

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = x_4 \quad \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix} = x_5 \quad \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} = x_6$$

Step 3: From elements of the lower part we construct another six-second ordered determinants and evaluate them.

$$\begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} = y_1 \quad \begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} = y_2 \quad \begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix} = y_3$$

$$\begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} = y_4 \quad \begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} = y_5 \quad \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} = y_6.$$

Step 4 : We write the values of these determinants in the structure by assigning +ve or -ve values as shown.

$$\begin{array}{cccccc} + & - & + & + & - & + \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{array}$$

Step 5 : Apply sutra { 1 } and then,

$$D = x_1 y_6 - x_2 y_5 + x_3 y_4 + x_4 y_3 - x_5 y_1 + x_6 y_1$$

Remark :- In solving examples, we apply row / column operation in such a way that we should get zeros in any one column or row of upper / left part, or, zeros in any one column or row of lower / right part. This will reduce three out of six determinants to zeros. The corresponding Tiryak (cross) products need not be evaluated as their values, by corresponding Tiryak method, will be zeros.

Ex. 7. Find the value of the determinant

$$D = \begin{vmatrix} 1 & 2 & 0 & 1 \\ 3 & 1 & 0 & 4 \\ 5 & 1 & -1 & 2 \\ 3 & 2 & 1 & 1 \end{vmatrix}$$

Step 1. Divide D by a horizontal line as shown.

Step 2. Observe that we have zeros in third column of upper part. We write the second order determinants and evaluate.

$$\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix} = -5 \quad \begin{vmatrix}1 & 0\\ 3 & 0\end{vmatrix} = 0 \quad \begin{vmatrix}1 & -1\\ 3 & 4\end{vmatrix} = 7$$

$$\begin{vmatrix}2 & 0\\ 1 & 0\end{vmatrix} = 0 \quad \begin{vmatrix}2 & -1\\ 1 & 4\end{vmatrix} = -9 \quad \begin{vmatrix}0 & -1\\ 0 & 4\end{vmatrix} = 0$$

$$\begin{vmatrix}5 & 1\\ 3 & 2\end{vmatrix} = 8 \quad \begin{vmatrix}5 & -1\\ 3\ 1 & = 8\end{vmatrix}\ 3 \quad \begin{vmatrix}5 & 2\\ 1 & = -1\end{vmatrix}$$

$$\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix} = 3 \quad \begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix} = -3 \quad \begin{vmatrix}-1 & 2\\ 1 & 1\end{vmatrix} = -3$$

Note:- we avoid the evaluation of first, third and fifth determinant as we have zeros at corresponding cross level.

Step 3 : We write the structure as :

+	-	+	+	-	+
-5	0	7	0	9	0
—	8	—	3	—	-3

Thus $D = (-5)(-3)+(7)(3)-(9)(8) = 15 + 21 - 72 = -36$

Ex. 8 Evaluate $D = \begin{vmatrix}1 & 2 & 3 & 8\\ 2 & 29 & 19 & 39\\ 3 & 15 & 16 & 33\\ 4 & 14 & 17 & 38\end{vmatrix}$

Step1 : - We apply row operations $R_2 - 2R_1, R_3 - 3R_1, R_4 - 4R_1$

Then D =

$$\begin{vmatrix} 1 & 2 & 3 & 8 \\ 0 & 25 & 13 & 23 \\ 0 & 9 & 7 & 9 \\ 0 & 6 & 5 & 6 \end{vmatrix}$$

Step 2 : Apply $c_2 - c_4$, then

$$D = \begin{vmatrix} 1 & -6 & 3 & 8 \\ 0 & 2 & 13 & 23 \\ 0 & 0 & 7 & 9 \\ 0 & 0 & 5 & 6 \end{vmatrix}$$

$$= (2)(42 - 45)$$

$$= -6$$

22.7 Solution of linear equations in two variables

Sutra लोपनस्थापनाभ्याम् (lopnasthapanabhyam)

(By alternate elimination and retention)

Method :- We write the equations as

$a_1 x + b_1 y = c_1$

$a_2 x + b_2 y = c_2$

Step 1 :— We eliminate x from the equations and write :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$= p$ $\qquad\qquad$ $= m$

Hence $p\,y = m$ $\qquad$ i.e. $y = (m / p)$

Step 2 :— We eliminate y from the equations and write :

$$\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$-p$ $\qquad\qquad$ $= n$

. Hence $-p\,x = n$ $\qquad$ i.e. $x = (-n / p)$.

Note : -

1] Above two methods are similar to Crammer's Rule for solutions of linear equations , and fail when $D = 0$.

2] Method 2 is further extended for solutions of linear equations in three and four variables,

3] We may apply row operations to reduce the determinants.

Ex. 9 Solve $3x - 2y = 4$, $x + y = 7$

Step

[1]. Eliminating x from the equations we get :-

$$\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} y = \begin{vmatrix} 3 & -4 \\ 1 & 7 \end{vmatrix}$$

Hence $5y = 25$ and $y = 5$

Step [2] Eliminating y from the equations we get : -

$$\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} x = \begin{vmatrix} -2 & -4 \\ 1 & 7 \end{vmatrix}$$

Hence $-5x = -10$ and $x = 2.$

22.8 Solutions of Linear Equations In Three Variables.

To solve the equations :—

$a_1 x + b_1 y + c_1 z = d_1$ (1)

$a_2 x + b_2 y + c_2 z = d_2$ (2)

$a_3 x + b_3 y + c_3 z = d_3$ (3)

Method : - This method is an extension of the method in previous case.

Step 1 :- We eliminate x from (1) and (2),

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y + \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} z = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

We get :$m_1 y + n_1 z = p$ (4).

Step 2 :- We eliminate x from (2) and (3),

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} y + \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} z = \begin{vmatrix} a_2 & d_2 \\ a_3 & d_3 \end{vmatrix}$$

We get : —$m_2 y + n_2 z = p_2$ (5).

Step 3:—

Solve equations (4) and (5) by the method or by inspection.

Let $y = y_1$ $z = z_1$ be the solutions.

Step 4 :-

We put these values in any suitable equation (1),or (2), or (3) to find the value of x.

Ex. 11:- Solve the equations :—

$2x + 3y - 4z = 3$ (1).

$x + 2y + z = 4$ (2).

$2x + y + 6z = 4$ (3).

Answer. Eliminating x from (1) and (2) we get :

$$\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} y + \begin{vmatrix} 2 & -4 \\ 1 & 1 \end{vmatrix} z = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

Hence $y + 6z = 5$ (4)

Now. Eliminating x from (2) and (3) we get :

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} y + \begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} z = \begin{vmatrix} 1 & 4 \\ 2 & 4 \end{vmatrix}$$

Hence $-3y + 4z = -4$ (5)

Now. eliminating y from (4) and (5) we get :

$$\begin{vmatrix} 1 & 6 \\ -3 & 4 \end{vmatrix} z = \begin{vmatrix} 1 & 5 \\ -3 & -4 \end{vmatrix}$$

Hence $\quad 22z = 11 \quad$ i.e. $\quad z = 1/2$

From (4) for $z = 1/2$, $y + 6(1/2) = 5 \; y = 2$

From (2) $\quad x = 4 - 4 - (1/2) = -1/2 \qquad x = -1/2.$

22.9 Solutions of Linear Equations in Four Variables.

To solve the equations :

$a_1 x + b_1 y + c_1 z + d_1 t = p_1$ (1)

$a_2 x + b_2 y + c_2 z + d_2 t = p_2$ (2)

$a_3 x + b_3 y + c_3 z + d_3 t = p_3$ (3)

$a_4 x + b_4 y + c_4 z + d_4 t = p_4$ (4)

Method :- This is an extension of the previous methods for two and three variables.

Step.1 :- Eliminate x and y from (1), (2) and (3)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} z + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} t = \begin{vmatrix} a_1 & b_1 & p_1 \\ a_2 & b_2 & p_2 \\ a_3 & b_3 & p_3 \end{vmatrix}$$

Observe that first two columns in these determinants are identical, hence expand the determinants by considering third column as base. For this apply Sutra{ 1 } to first two columns we get

$$a_1 b_2 - a_2 b_1 = e_1 \quad \text{(say)}$$
$$a_1 b_3 - a_3 b_1 = e_2$$
$$a_2 b_3 - a_3 b_2 = e_3$$

Now we write the structure as follows:

$$\begin{array}{llll} e_1 & c_1 & d_1 & p_1 \\ e_2 & c_2 & d_2 & p_2 \\ e_3 & c_3 & d_3 & p_3 \end{array}$$

Again apply Sutra { 1 } we get

$$e_1 c_3 - e_2 c_2 + e_3 c_1 = f_1 \quad \text{(say)}$$
$$e_1 d_3 - e_2 d_2 + e_3 d_1 = f_2$$
$$e_1 p_3 - e_2 p_2 + e_3 p_1 = f_3$$

Thus we get the first equation

$$f1\, z + f2\, t = f3 \qquad (5).$$

Step. 3 :- Eliminate x and y from (1), (2) and (4) we get

$$\begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a4 & b4 & c4 \end{vmatrix} z + \begin{vmatrix} a1 & b1 & d1 \\ a2 & b2 & d2 \\ a4 & b4 & d4 \end{vmatrix} t = \begin{vmatrix} a_1 & b_1 & p_1 \\ a_2 & b_2 & p_2 \\ a_4 & b_4 & p_4 \end{vmatrix}$$

Observe that first two columns in these determinants are identical, hence expand the determinants by considering third column as base. For this apply Sutra{ 1 } to first two columns we get

$$a_1 b_2 - a_2 b_1 = e_1 \quad \text{(say)}$$
$$a_1 b_4 - a_4 b_1 = e_4$$

$$a_1 b_2 - a_4 b_2 = e_5$$

Now we write the structure as follows:

$$\begin{array}{c|ccc} e_1 & c_1 & d_1 & p_1 \\ e_4 & c_2 & d_2 & p_2 \\ e_5 & c_4 & d_4 & p_4 \end{array}$$

Again apply Sutra { 1 } we get

$$e_1 c_4 - e_4 c_2 + e_5 c_1 = f_4 \qquad \text{(say)}$$
$$e_1 d_4 - e_4 d_2 + e_5 d_1 = f_5$$
$$e_1 p_4 - e_4 p_2 + e_5 p_1 = f_6$$

Thus we get the second equation

$$f4\, z + f5\, t = f6 \qquad (6).$$

Step [3] Solve the equations (5) and (6) by any suitable method. Let $z = z'$ $t = t'$ are the solutions .

Step [4]:-

We substitute these values in any two suitable equations (1) , (2) , (3) or (4) and solve them for x and y.

Ex.12:- Solve the equations :—

$$x - y - t = 0 \qquad (1)$$
$$x + y + z + 2t = 2 \qquad (2)$$
$$3x - y + 2z - t = 6 \qquad (3)$$
$$x + 3y + 2z + 3t = 10 \qquad (4)$$

Step.[1] We eliminate x and y from (1), (2) and (3)

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \\ 3 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 4 \\ 3 & -1 & 6 \end{vmatrix}$$

Apply Sutra { 1 } to first two identical columns we get

$2 = e_1 \qquad 2 = e_2 \qquad -4 = e_3$

Now we write the structure as follows:

$$\begin{array}{ccccc} 2 & 0 & 1 & 0 & \\ 2 & & 1 & 2 & 4 \\ -4 & & 2 & -1 & 6 \end{array}$$

Hence$f_1 = 4 - 2 = 2$, $f_2 = -2 - 4 + 4 = -2$, $f_3 = 12 - 8 = 4$.

Thus we get$2z - 2t = 4$ i.e. $z - t = 2$ (5)

Step.2 :- We eliminate x and y from (1), (2) and (4)

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} z + \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{vmatrix} t = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 4 \\ 1 & 3 & 10 \end{vmatrix}$$

Apply Sutra { 1 } to first two columns we get:

$e_1 = 2, \qquad e_4 = 4$ and $\qquad e_5 = 2$

We write the structure as

$$\begin{array}{cccc} 2 & 0 & -1 & 0 \\ 4 & 1 & 2 & 4 \\ 2 & 2 & 3 & 10 \end{array}$$

Thus $f_4 = 0$, $f_5 = -4$ and $f_6 = 4$

Hence $0z - 4t = 4$ i.e. $t = -1$.

Step. 3:- From 5 we get $z = 1$.

$(5) - (6)$ gives $z = 4 - 3$. Hence $z = 1$

And from (5) we get $3 - t = 4$. Hence $t = -1$

Step. 4:- For $z = 1$ and $t = -1$, in (1) and (2) we get

$x - y = -1$ and $x + y = 5$.

solving we get, $x = 2$ and $y = 3$.

Thus solutions are $x = 2, y = 3, z = 1$, and $t = -1$

22.10 Consistency of Three Equations.

Let $a_1x + b_1y = c$

$a_2x + b_2y = c_2$

$a_3x + b_3y = c_3$

These three equations are consistent iff

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(The proof is omitted)

Ex. 13:- Verify that the equations $3x - 4y = 10$, $2x + 3y = 1$, $x + y = 1$

are consistent and solve them.

Answer : Here $D = \begin{vmatrix} 3 & -4 & 10 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} 3 & 4 & 10 \\ -1 & 1 & 2 \end{vmatrix}$$

Hence D = $6 + 4 - 10 = 0$.

Hence the given equations are consistent.

We choose the equations

$2x + 3y = 1$

and $x + y = 1$

Eliminating x we get

$$\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \quad y = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

Hence $-y = 1$ i.e. $y = -1$ and then $x = 2$.

Ex. 14:- Find 'K' if the equations $Kx - 4y + 3 = 0$, $x + 3y - 7 = 0$, and $x + y = 3$ are consistent. Hence solve them.

Answer :-

We write the given equations as :-

$$\begin{aligned} Kx - 4y &= -3 \\ x + 3y &= 7 \\ x + y &= 3 \end{aligned}$$

and construct the determinant D as

$$D = \begin{vmatrix} K & -4 & -3 \\ 1 & 3 & 7 \\ 1 & 1 & 3 \end{vmatrix}$$

$$\Rightarrow \begin{matrix} K & 4 & -3 \\ -2 & -4 & 2 \end{matrix}$$

OR $D = 2K + (-16) + 6$

$D = 2K - 10$

The equations are consistent, hence $D = 0$

Thus $2K - 10 = 0$ hence $K = 5$

We solve the equations : $x + 3y = 7$ and $x + y = 3$

subtracting, we get $2y = 4$ i.e. $y = 2$ and $x = 1$

Ex.15:- Show that the equations $11x - y + 2z = 12$,

$3x + 2y - 4z = 1, x + 2y - z = 2$, and $7x - 5y - z = 4$ are consistent.

Answer:-

We construct the determinant :—

$$D = \begin{vmatrix} 11 & -1 & 2 & 12 \\ 3 & 2 & -4 & 1 \\ 1 & 2 & -1 & 2 \\ 7 & -5 & 2 & 4 \end{vmatrix}$$

Apply $C_3 + 2C_2$

$$D = \begin{vmatrix} 11 & -1 & 0 & 2 \\ 3 & 2 & 0 & 1 \\ 1 & 2 & 3 & 2 \\ 7 & -5 & -8 & 4 \end{vmatrix}$$

Apply $C_1 - C_4 - C_2$

$$D = \begin{vmatrix} 0 & -1 & 0 & 12 \\ 0 & 2 & 0 & 1 \\ -3 & 2 & 3 & 2 \\ 8 & -5 & -8 & 4 \end{vmatrix}$$

$$\Rightarrow \begin{matrix} + & - & + & + & - & + \\ 0 & 0 & 0 & 0 & -25 & 0 \\ — & 0 & — & — & — & — \end{matrix}$$

$\Rightarrow$ D = 0. Hence equations are consistent.

22.11 Applications of Determinants

(1). Pair Of Lines

We know that any second-degree equation of the type

$a x^2 + 2h x y + b y^2 + 2g x + 2f y + c = 0$ represents a pair of the lines (joint equation of the lines) if the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Now this determinant can be evaluated easily by above methods.

Ex. 16. Show that the equation $2x^2 + xy - y^2 + x + 4y - 3 = 0$ represents a pair of the lines.

Answer: - Comparing with second degree equation above, we get

$a = 2$, $b = -1$, $2h = 1$, $2g = 1$, $2f = 4$, $c = -3$,

thus $h = 1/2$, $g = 1/2$, $f = 2$.

Now $$D = \begin{vmatrix} 2 & 1/2 & 1/2 \\ 1/2 & -1 & 2 \\ 1/2 & 2 & -3. \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 & 1 \\ 1/2 & -1 & 2 \\ 1/2 & 2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

Expanding the determinant we get

$$4D = \begin{matrix} 8 & -1 & 1 \\ 3 & -5 & -1 \end{matrix}$$

$$= 8(-1) - (1)(-5) + (1)(3) = 0$$

$$\text{Hence } D = 0$$

Hence the equation represents a pair of the lines.

(2) Product of the Vectors

Let $\mathbf{a} = a_1 i + a_2 j + a_3 k$, $b = \mathbf{b}_1 i + b_2 j + b_3 k$, and $\mathbf{c} = c_1 i + c_2 j + c_3 k$ are any three vectors.

then the vector product or cross product of **a** and **b** is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The scalar triple product of **a** , **b**, **c** is defined as :-

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a1 & a2 & a3 \\ b1 & b2 & b3 \\ c1 & c2 & c3 \end{vmatrix}$$

Above determinant can be evaluated by VM methods easily.

EXERCISE [1]

Evaluate following determinants . Apply elementary operations if necessary.

Set A :-

[1] $\begin{vmatrix} -1 & 3 \\ 5 & 2 \end{vmatrix}$ [2] $\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix}$ [3] $\begin{vmatrix} x & 3 \\ y & 4 \end{vmatrix}$ [4] $\begin{vmatrix} a+7 & b \\ a-1 & b+2 \end{vmatrix}$

Set B :-Find the value of x if :-

[1] $\begin{vmatrix} 2 & x \\ 3 & 5 \end{vmatrix} = 3$ [2] $\begin{vmatrix} x+1 & x-1 \\ 5 & 6 \end{vmatrix} = 0$ [3] $\begin{vmatrix} 3x-1 & -2 \\ 0 & 1-x \end{vmatrix} = 0$

Set C :-

[1] $\begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix}$ [2] $\begin{vmatrix} -1 & -2 & 3 \\ 4 & -5 & 2 \\ -2 & 1 & 1 \end{vmatrix}$ [3] $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix}$

[4] $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 3 & 1 & 4 \end{vmatrix}$ [5] $\begin{vmatrix} -7 & 1 & 4 \\ -3 & 3 & 0 \\ 8 & -2 & -2 \end{vmatrix}$ [6] $\begin{vmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & 18 & 19 \end{vmatrix}$

[7] $\begin{vmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{vmatrix}$ [8] $\begin{vmatrix} 2 & 6 & 7 \\ 5 & 4 & 8 \\ 3 & 1 & 9 \end{vmatrix}$

<u>*Set D*</u> :-

[1] $\begin{vmatrix} 1 & -3 & 4 & 5 \\ 2 & 1 & 3 & 4 \\ -1 & 2 & 3 & 1 \\ 1 & 0 & -1 & 3 \end{vmatrix}$ [2] $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 34 \end{vmatrix}$

[3] $\begin{vmatrix} 6 & 1 & 3 & 2 \\ 5 & 0 & 3 & 4 \\ 1 & 3 & 4 & 2 \\ 5 & 1 & 2 & 3 \end{vmatrix}$ [4] $\begin{vmatrix} 2 & 5 & 7 & 8 \\ 3 & 6 & 2 & 4 \\ 3 & 5 & 7 & 6 \\ 1 & 6 & 9 & 7 \end{vmatrix}$

ANSWERWERS [1]

Set A :

[1] -17 [2] 6 [3] $4x - 3y$ [4] $2a + 8b - 14$

Set B:

[1] $x = 7/3$ [2] $x = -11$ [3] $x = 1, 1/3$ [4] $x = -2, 3/2$.

Set C:

[1] D = 1 [2] D = - 5 [3] D = - 7 [4] D = - 45

[5] D = - 36 [6] D = 0 [7] D = 0 [8] D = 0.

Set D:

[1] D = 128 [2] D = 0 [3] D = - 66 [4] D = 186.

EXERCISE [2]

Solve the following linear equations using determinants :-

<u>**Set A**</u> :-

[1] $x + 3y = 7$, $2x - y = -7$. [2] $3x + 8y = 0$, $2x - 3y = 5$

[3] $2x - 5y = 7$, $6x + 3y = 3$ [4] $5x + 7y = 35$, $3x + 4y = 6$.

Set B :-

[1] $x - y + z = 2$, $2x + y + z = 3$, $x + 4y - 3z = -9$.

[2] $2x - 2y + 3z = 4$, $x + 4y - 2z = 10$, $4x - 3y + z = 1$.

[3] $x + y - z = -4$, $2x + y - z = -3$, $x - y - z + 12 = 0$.

[4] $2x + 4y + z = 13$, $-x + y + 3z = 11$, $5x - 3y - z = 15$.

[5] $x + 2y + 3z = 12$, $2x + 3y + 4z = 18$, $4x + 3y + 5z = 24$

Set C :-

[1] $x + y - z + t = 4$, $6x + 3y + 4z - 3t = 12$,

$2x - y + 2z - 2t = 6$, $3x + 2y = 3z - 4t = 0$.

[2] $x + 2y + 2z + t = -5$, $2x - y + 6z + 3t = 8$,

$x + y + z - t = 3$ $2x - 3y + 2z = -4$.

[3] $3x - 2y - w = 2$, $2y + 2z + w = 1$,

$x - 3y - 3z + 2w = 3$, $y + 2z + w = 1$.

ANSWERWERS [2]

Set A :-

[1] $x = -2$, $y = 3$. [2] $x = 8/15$, $y = -1/5$.

[3] $x = 1$, $y = -1$. [4] $x = -98$, $y = 75$.

Set B :-

[1] $x = 1$, $y = -1$, $z = 2$. [2] $x = 2$, $y = 3$, $z = 2$.

[3] $x = 1$, $y = 4$, $z = 9$. [4] $x = 4$, $y = 0$, $z = 5$.

[5] $x = -6$, $y = 18$, $z = -6$.

Set C :-

[1] $x = 4, \ y = -2, z = 0. t = 2.$

[2] $x = -1, \ y = 2, z = 2. t = 0.$

[3] $x = 1, \ y = 0, z = 0. w = 1.$

EXERCISE [3]

Set A :-

Show that the following equations are consistent and solve them.

[1] $2x + 3y = 9, 4x - y = 11, \ 2x + y = 7.$

[2] $3x + 21y = 24, 6x + 7y = 28, 13x + 7y = 8.$

[3] $4x - 3y + 1 = 0, \ x + y = 5, 2x - 3y + 5 = 0, -2x + y + 1 = 0$

Set B :-

Find K if following equations are consistent and solve them.

[1] $3x + Ky = 7, 2x + 3y = 1, x + y = 6.$

[2] $5x + 5y + z = K, \qquad x + y + z = 3,$

$3x - 2y + 3z = 4, \qquad 2x - y + 2z = 3.$

ANSWERWERS [3]

Set A :-

[1] $x = 3, y = 1.$ [2] $x = 0, y = 8/7.$ [3] $x = 2, y = 3$

Set B :-

[1] $K = 2, x = 19/5, y = 11/5.$ [2] $K = 11, x = y = z = 1.$

EXERCISE [4]

1) Show that following equations represents a pair of the lines.

[1] $9x^2 - y^2 + 3x + y = 0$,

[2] $2x^2 + 7xy + 3y^2 - 5x - 5y + 2 = 0$

2) Find K if the following equations represent a pair of lines.

[1] $Kxy + 10x + 6y + 4 = 0$, [2] $2x^2 + xy - y^2 + x + 4y + K = 0$

ANSWERWERS [4]

[1] $K = -6$ [2] $K = -3$.

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CHAPTER 23.

PARTIAL FRACTIONS

23.1 We often require the process of separating a given rational expression into a group of simpler fractions, called partial fractions. In addition to most general method of equating the coefficients, various other methods are discussed in textbooks of higher algebra and integral calculus.

The method suggested by VM for resolution of rational fraction with denominator as distinct linear or quadratic factors is similar to current method on many counts. However, Swamiji has suggested the use of differential calculus when rational fraction has repeated factors.

23.2 Here this idea is further extended in other cases to find partial fraction in a less laborious manner. Now we enlist the prerequisites for this extended method.

[1] The function $F(x) = \{ f(x) / g(x) \}$, where $f(x)$ and $g(x)$, $g(x)$ 0 are polynomials is called rational fraction (R. F) and every rational fraction can be expressed as partial fractions whose denominators are of the form

$$(ax + b)^n \text{ and } (ax^2 + bx + c) \text{ etc.}$$

[2] If two polynomials of same degree are equal for all values of the variable, coefficients of two like powers of the variable in the two polynomials are equal,

[3] If $y = x^n$ then $y_1 = (dy / dx) = n x^{n-1}$ and

if $y = \{ f(x) \}^n$ then $y_1 = n \{ f(x)^{n-1} \} f_1(x)$, etc.

[4] Readers are suggested to study the methods of finding derivatives ofproduct of functions discussed earlier.

[5] The equations $a_1 x + b_1 y = c_1$, and $a_2 x + b_2 y = c_2$ are either

solved by inspection or any other suitable method including use of determinants.

[6] The value of $f(x).(x-a)^m$ is zero for $x = a$, and n' Th derivative of $f(x).(x-a)^m$ is also zero for $n < m$ for $x = a$.

[7] Finally a few examples are solved by the mixture of VM method and current methods to facilitate the solutions quickly.

Illustrative Examples

TYPE 1. Distinct Linear or Quadratic Factors: -

Ex. 1: - Resolve into partial fractions: —

$$\frac{2x+5}{(x+1)(x-4)(2x-3)}.$$

Let $$\frac{2x+5}{(x+1)(x-4)(2x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-4)} + \frac{C}{(2x-3)}$$

Where A , B , C are real constants, then,

$$N(x) = A(x-4)(2x-3) + B(x+1)(2x-3) +$$

$$+ C(x+1)(x-4)$$

And $N(x) = 2x + 5$.

1] Put $x + 1 = 0$, i.e. $x - 1$ in $N(x)$ Sutra { 1 }.

Hence $N(-1) = A(-5)(-5)$

Thus $3 = A(25)$ $A = (3/25)$.

2] Put $x - 4 = 0$, i.e. $x = 4$ in $N(x)$,

Hence $N(4) = B(5)(5)$

Thus $13 = B(25)$ $B = (13/25)$.

3] Put $2x - 3 = 0$, i.e. $x = (3/2)$in $N(x)$,

Thus $8 = C(5/2)(-5/2)$ $C = -(32/25)$.

Hence partial fractions are :-

$$\text{R. F.} = \frac{3}{25(x+1)} + \frac{13}{25(x-4)} - \frac{32}{25(2x-3)}$$

Ex. 2 :— Resolve into partial fractions :—

$$\frac{2x^3 + 7x^2 + 1}{(x^2 - x - 1)(x^2 + 2x + 3)}$$

$$\text{Let } \frac{2x^3 + 7x^2 + 1}{(x^2 - x - 1)(x^2 + 2x + 3)} = \frac{Ax + B}{(x^2 - x - 1)} + \frac{Cx + D}{(x^2 + 2x + 3)}$$

Then $N(x) = (Ax + B)(x^2 + 2x + 3) + (Cx + D)(x^2 - x - 1)$

Where $N(x) = 2x^3 + 7x^2 + 1$.

1] Put $x^2 - x - 1 = 0$ i.e. $x^2 = x + 1$, $x^3 = x^2 + x = 2x + 1$,

$N(x) = 4x + 2 + 7x + 7 + 1 = 11x + 10$

and $x^2 + 2x + 3 = x + 1 + 2x + 3 = 3x + 4$.

We get $11x + 10 = (Ax + B)(3x + 4)$

$= 3Ax^2 + (4A + 3B)x + 4B$

$= 3A(x+1) + (4A+3B)x + 4B$

$= (7A + 3B)x + (3A + 4B)$

Comparing, we get $7A + 3B = 11$ ——————— [1]

$3A + 4B = 10$ ——————— [2]

Eliminating A from [1] and [2] we get

$$\begin{vmatrix} 7 & 3 \\ 3 & 4 \end{vmatrix} B = \begin{vmatrix} 7 & 11 \\ 3 & 10 \end{vmatrix}$$

i.e. $19B = 37$ hence $B = (37/19)$

Now eliminating B from [1[and [2] we get

$$\begin{vmatrix} 3 & 7 \\ 4 & 3 \end{vmatrix} A = \begin{vmatrix} 3 & 11 \\ 4 & 10 \end{vmatrix}$$

i.e. $-19A = -14$ hence $A = 914/19)$.

2] Now put $x = 0$ in $N(x)$ we get, $N(0) = B(3) + D(-1)$

$1 = (112/19) - D$

Hence $D = (92/19)$

3] Comparing coefficients of x^3 in $N(x)$ we get

$2 = A + C$, Hence $C = (24/19)$

Hence partial fractions are :-

$$R.F. = \frac{14x + 3^7}{1^{r}(x^2 - 2x - 1)} + \frac{24x + 92}{19(x^2 + 2x + 3)}$$

Note :- Readers are suggested to compare this method with current method of solving the same problem.

TYPE 2. Repeated Linear Factors :-

Ex. 3 :- Resolve $\dfrac{x^3+2x^2-x+5}{(2x+1)^2(1-x)^2}$ into partial fractions

Ans.:- let this be equal to :

$$\frac{A}{2x+1}+\frac{B}{(2x+1)^2}+\frac{C}{1-x}+\frac{D}{(1-x)^2}.$$

Then $N(x) = [A(2x+1)+B](1-x)^2+$
$+[C(1-x)+D](2x+1)^2,$

where $N(x) = x^3+2x^2-x+5.$

1] For $1-x = 0$ i.e. $x = 1$ we observe that

$[A(2x+1)+B](1-x)^2$ reduces to zero along with its first order derivative.

Now $N_1(x) = 3x^2+4x-1$, Thus $N(1) = 7$, $N_1(1) = 6$

We write the structure as : —

$u = C(1 - x) + D$ $\qquad$ $u_1 = -C$

$= D$ $\qquad$ $= -C$ $\qquad$ For $x = 1$

$v = (2x + 1)^2$ $\qquad$ $v_1 = 4(2x + 1)$

$= 9$ $\qquad$ $= 12$ $\qquad$ For $x = 1$.

For $x = 1$ we get : $N(1) = 9D$, Hence $D = (7/9)$

Values of first order derivatives of $N(x)$ for $x = 1$ gives

$$N_1(1) = D(12) + 9(-C)$$

$$6 = (28/3) - 9C \text{ Hence } C = (10/27)$$

2] For $2x + 1 = 0$ i.e. $x = -1/2$ we observe that

$[C(1 - x) + D](2x + 1)^2$ reduces to zero along with its first order derivative.

Now $N(-1/2) = 47/8$ and $N_1(-1/2) = -9/4$

We write the structure as : —

$u = A(2x + 1) + B$ $\qquad$ $u = 2A$

$= B$ $\qquad$ $= 2A$ $\qquad$ For $x = -1/2$

$v = (1 - x)^2$ $\qquad$ $v = -2(1 - x)$

$= 9/4$ $\qquad$ $= -3$ For $x = -1/2$

For $x = -1/2$ in $N(x)$ we get (Urdhva)

$$N(-1/2) = B(9/4)$$

$$(47/8) = B(9/4) \text{ Hence } B = (47/18)$$

Values of first order derivatives of $N(x)$ for $x = -1/2$ gives

$N_1(-1/2) = B(-3) + 2A(9/4)$ $\qquad$ (Tiryak)

i. e. $(9/2)A = (47/4) - (9/4)$ Hence $A = (67/54)$

Hence partial fractions are :-

$$R.F. = \frac{67}{54(2x+1)} + \frac{47}{18(2x+1)^2} + \frac{10}{27(1-x)} + \frac{7}{9(1-x)^2}$$

Ex. 4 Resolve $\dfrac{x^3}{(x-1)^4(x^2-x+1)}$ into partial fractions

Ans :- Let R. F. is :-

$$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4} + \frac{Ex+F}{(x^2-x+1)}$$

$$N(x) = [A(x-1)^3 + B(x-1)^2 + C(x-1) + D](x^2-x+1)$$
$$+ (Ex+F)(x-1)^4$$

where $N(x) = x^3$

1] We observe that the term $(Ex + F)(x-1)^4$ reduces to zero along with its first, second third order derivatives for $x = 1$. Now

$N(x) = x^3$ $N_1(x) = 3x^2$ $N_2(x) = 6x$ $N_3(x) = 6$

$= 1$ $= 3$ $= 6$ $= 6$

For x=1

We write the structure as : —

$u = D$ $u_1 = C$ $u_2 = 2B$ $u_3 = 6A$.

$v = x^2 - x + 1$ $v_1 = 2x - 1$ $v_2 = 2$ $v_3 = 0$

$= 1$ $= 1$ $= 2$ $= 0$

For x = 1.

Values of $N(x)$ for $x = 1$ gives (Urdhva)

$N(1) = D$ Hence $D = 1$

Values of first order derivatives of $N(x)$ for $x = 1$ gives

$N_1(1) = D(1) + C(1)$ (First tiryak)

$3 = 1 + C$ Hence $C = 2$.

Values of second order derivatives of $N(x)$ for $x = 1$ gives

$N_2(1) = 2D + \{2\} C + 2B$(second Tiryak)

$6 = 2 + 4 + 2B$

Hence $B = 0$.

Values of third order derivatives of $N(x)$ for $x = 1$ gives

$N_3(1) = 0 + \{3\} 2C + \{3\} 3B + 6A$ (Third Tiryak)

$6 = 0 + 12 + 0 + 6A$

Hence $A = -1$.

2] Values of $N(x)$ for $x = 0$ gives

$N(0) = [-A + B - C + D](1) + F(1)$

$0 = 1 - 2 + 1 + F$ Hence $F = 0$

3] Comparing coefficients of x^5 in $N(x)$ we get

$0 = A + E$

Hence $E = 1$.

Hence partial fractions are :-

$$\frac{-1}{x-1} + \frac{2}{(x-1)^3} + \frac{1}{(x-1)^3} + \frac{x}{x^2-x+1}$$

TYPE 3. Repeated Quadratic Factors :-

Ex. 5 Resolve $\dfrac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}$ into partial fractions.

Ans :- Let R. F. be :-

$$\frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3}$$

$N(x) = (Ax + B)(x^2 + 2)^2 + (Cx + D)(x^2 + 2) + (Ex + F)$,

where $N(x) = x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4$

1] For $x^2 - 2 = 0$ i.e. . $x^2 = -2$,

we get $x^3 = -2x^2 = 4$, $x^5 = 4$

Hence $N(x) = 4x - 4 - 8x + 8 + 8x - 4 = 4x$

And $N_1(x) = 4$

We write the structure as :-

$u = Cx + D$ $u_1 = C$ $w = Ex + F$

$+$

$v = x^2 + 2$ $v_1 = 2x$ $w_1 = E$

Note:- The term $(Ax + B)(x^2 + 2)^2$ reduces to zero along

with its first order derivative for $x^2 = -2$.

Values of $N(x)$ for $x^2 = -2$ gives (Urdhva)

$$4x = (Cx + D)(0) + Ex + F$$

$$4x + 0 = Ex + F$$

Comparing we get $E = 4, F = 0.$

Values of first order derivatives of $N(x)$ for $x^2 = 1$ gives

$$N_1(x) = (Cx + D)(2x) + C(0) + E$$

$$4 \quad = 2Cx^2 + 2Dx + E$$

$$4 \quad = -4C + 2Dx + 4$$

Hence $0 = -4C + 2Dx$

Comparing we get :- $C = 0$ $D = 0.$

2] Values of $N(x)$ for $x = 0$ gives

$-4 = 4B + 2D + F$ Hence $B = -1.$

3] Comparing coefficients of x^5 in $N(x)$ we get

$1 = A + C$ Hence $A = 1.$

Hence partial fractions are :-

$$\text{R.F.} \quad \frac{x-1}{(x^2+2)} + \frac{4x}{(x^2+2)^3}.$$

TYPE 4. Higher Degree Factors :-

Ex. 6 :- Resolve

$$\frac{x^4 - 3x^2 + 4}{(x-1)^2 (x^3 + 2x + 1)}$$

into partial fractions.

Ans :- Let R. F. be :-

$$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx^2 + Dx + E}{(x^3 + 2x + 1)}$$

$N(x) = [A(x-1+B](x^3 + 2x + 1) + (Cx^2 + Dx + E)(x-1)^2$

Where $N(x) = x^4 - 3x^2 + 4$.

1] The term $(Cx^2 + Dx + E)(x-1)^2$ reduces to zero along with its first order derivative for $x = 1$.

Now $N(1) = 2$. $N_1(x) = 4x^3 - 6x$ i.e. $N_1(1) = -2$

We write the structure as :-

$u = A(x-1) + B$ $\qquad u_1 = A$

$v = (x^3 + 2x + 1)$ $v_1 = 3x^2 + 2$

$= 4$ $\qquad\qquad = 5$

For $x = 1$.

Values of $N(x)$ for $x = 1$ gives (Urdhva)

$N(1) = (B)(4)$ $\qquad$ Hence $B = 1/2$.

Values of first order derivatives of $N(x)$ for $x = 1$ gives

$N_1(1) = 5B + 4A$

$2 \quad = (5/2) + 4A$ $\qquad$ Hence $A = -9/8$.

2] For $x = 0$ we get $N(0) = 4$, $N_1(0) = 0$.

We write the structure as :-

$u = A(x-1) + B$ $\quad$ $u_1 = A$

$= -A + B = (13/8)$ $\quad$ $= -9/8$

$v = (x^3 + 2x + 1)$ $\quad$ $v_1 = 3x^2 + 2$

$= 1$ $\quad$ $= 2$ For $x =$

$w = (Cx^2 + Dx + E)$ $\quad$ $w_1 = 2Cx + D$

$= E$ $\quad$ $= D$

$z = (x-1)^2$ $\quad$ $z_1 = 2(x-1)$ For $x = 0$.

Values of $N(x)$ for $x = 0$ gives (Urdhva)

$$N(0) = (13/8)(1) + E(1)$$

$$4 = (13/8) + E \text{ Hence } E = 19/8.$$

Values of first order derivatives of $N(x)$ for $x = 0$ gives

$$N_1(0) = (13/8)(2) - (9/8) + E(-2) + D(1)$$

$$0 = (13/4) - (9/8) - (19/4) + D$$

Hence $D = 21/8$.

3] Comparing coefficients of x^4 in $N(x)$ we get

$1 = A + C$ $\quad$ Hence $C = 17/8$.

Hence partial fractions are :-

$$\text{R.F.} = \frac{9}{8(x-1)} + \frac{1}{2(x-1)^2} + \frac{17x^2 + 21x + 19}{8(x^3 + 2x + 1)}$$

TYPE 5. Factors of the type x^n

Ex. 7:- Resolve $\dfrac{1 + 2x}{x^2 (x+2)^3 (x-1)}$ into partial factors.

Ans :- Let R. F. be

$$\frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x+2} + \frac{E}{(x+2)^2} + \frac{F}{(x+2)^3}$$

$N(x) = x^2 (x+2)^3 + (Bx + C)(x-1)(x+2)^3 +$

$+ [D(x+2)^2 + C(x+2) + F] x^2 (x-1),$

Where $N(x) = 1 + 2x$.

1) For $x = 1$ in $N(x)$, we get

$N(1) = A(1)^2 (1+2)^3$

$3 = 27A$ Hence $A = 1/9$.

2) For $x = 0$ in $N(x)$, we note that $Ax^2 + (x+2)^3$ and $[D(x+2)^2 + E(x+2) + F] x^2 (x-1)$ reduces to zero along with its first order derivative.

We write the structure as :-

$u = Bx + C$ $\qquad u_1 = B$

$= C$ $\qquad = B$ For $x = 0$

$v = x - 1$ $\qquad v_1 = 1$

$= -1$ $\qquad = 1$

For $x = 0$

$w = (x+2)^3$ $\qquad\qquad$ $w_1 = 3(x+2)^2$

$= 8$ $\qquad\qquad$ $= 12$ For $x = 0$

Values of $N(x)$ for $x = 0$ gives (Urdhva)

$N(0) = C(-1)(8)$ Hence $C = (-1/8)$

Values of first order derivatives of $N(x)$ for $x = 0$ gives

$N_1(0) = B(-1)(8) + 8C + 12C(-1)$

$2 \quad = -8B - 1 + (3/2)$ Hence $B = -3/16$.

3) For $x+2 = 0$ i. e. $x = -2$ the first and second term of $N(x)$ reduces to zero along with its first order derivatives.

Now Values of $N(x)$ for $x = 0$ gives (Urdhva)

$N(-2) = -3$ and $N_1(-2) = 2$ $\qquad$ $N_2(-2) = 0$

We write the structure as :-

$u = F \quad u_1 = E \quad u_2 = 2D$ $\qquad$ for $x = -2$

$v = -12 \quad v_1 = 16 \quad v_2 = -14$ $\qquad$ for $x = 0$

Values of $N(x)$ for $x = -2$ gives (Urdhva)

$N(-2) = -12F$ $\qquad$ Hence $F = 1/4$

Values of first order derivatives of $N(x)$ for $x = -2$ gives

$N_1(-2) = 16F - 12E$

$2 = 4 - 12E$ $\qquad$ Hence $E = 1/6$.

Values of second order derivatives of $N(x)$ for $x = -2$ gives

$$N_2(-2) = -14F + 32E - 24D$$

$$0 = (-7/2) + (16/3) - 24D$$

Hence $D = (11/144)$.

Hence partial fractions are :-

$$\text{R.F. } \frac{1}{9(x-1)} + \frac{-3}{16x} + \frac{1}{8x^2} + \frac{11}{144(x+2)} +$$

$$+ \frac{1}{6(x+2)^2} + \frac{1}{4(x+2)^3}$$

Ex. 8:-Resolve $\dfrac{x^5+1}{x^3(x^3+x^2-x-1)}$ into partial fractions

Ans :- Let R. F. be

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx^2+Ex+F}{(x^3+x^2-x-1)}$$

$N(x) = (Ax^2+Bx+C)(x^3+x^2-x-1) + (Dx^2+Ex+F)x^3$,

Where $N(x) = x^5 + 1$

Now $N(x) = 5x^4$, $N(x) = 20x^3$

1) For $x = 0$, $(Dx^2+Ex+F)x^3$ reduces to zero along with its first order derivatives.

We write the structure as :- For x = 0

$u = (A x^2 + B x + C)$ $\quad u = 2 A x + B$ $\quad u = 2 A$

$= C$ $\quad = B$ $\quad = 2 A$

$v = (x^3 + x^2 - x - 1)$ $\quad v = 3 x^2 + 2 x - 1$ $\quad v = 6 x + 2$

$= -1$ $\quad = -1$ $\quad = 2$

Values of N (x) for x = 0 gives (Urdhva)

$$N(0) = C(-1)$$

$1 = C(-1)$ Hence $C = -1$

Values of first order derivatives of N (x) for x = 0 gives

$$N(0) = C(-1) + B(-1)$$

$$0 = 1 - B$$

Hence B = 1

Values of second order derivatives of N (x) for x = 0 gives

$$N(0) = C(2) + 2B(-1) + 2A(-1)$$

$0 = -2 - 2 - 2A$ Hence $A = -2$

2) Comparing coefficients of x^5 in N (x) we get

$1 = A + D$ Hence $D = 3$

3) Comparing coefficients of x^4 in N (x) we get

$0 = A + B + E$ Hence $E = 1$

4) Comparing coefficients of x^3 in N (x) we get

$0 - A + B + C + F$ Hence $F = -2$

Hence partial fractions are :-

$$R.F = \frac{-2}{x} + \frac{1}{x^2} + \frac{-1}{x^3} + \frac{3x^2 + x - 2}{(x^3 + x^2 - x - 1)}$$

EXERCISE

Resolve the following rational fractions into partial fractions :

[1] $$\frac{x^2 - 3x - 1}{x\,(x-1)\,(x+2)}$$

[2] $$\frac{21x - 23}{x^2 - 7x + 12}$$

[3] $$\frac{x^3 + x^2 + x + 2}{(x^2+1)\,(x^2+2)}$$

[4] $$\frac{6x^4 + 11x^3 + 18x^2 + 14x + 6}{(x+1)\,(x^2+x+1)^2}$$

[5] $$\frac{9x^3 - 24x^2 + 48x}{(x+1)(x-2)^4}$$

[6] $$\frac{3x^3 - 8x^2 + 10}{(x-1)^4}$$

(282)

[7] $$\frac{4x^4 - 16x^3 + 17x^2 - 8x + 7}{(x-1)(x-2)^2(x^2+1)}$$

8] $$\frac{x^4}{(x^2+1)^2(x^2+3)^3}$$

[9] $$\frac{x^3 + 2x^2 + x - 2}{(x+2)(x^4 + 3x^3 - x^2 + 4)}$$

[10] $$\frac{x^4 + 8x^3 - x^2 + 2x + 1}{x(x+1)^2(x^2 - x + 1)}$$

[11] $$\frac{x^4 + x^3 + 8x^2 + x + 2}{x^3(x^2+1)^3}$$

[12] $$\frac{x-5}{(2x+1)(2x+5)}$$

ANSWERS

1] $$\frac{1}{2x} + \frac{-1}{x-1} + \frac{-3}{2(x+2)}$$

2] $$\frac{61}{(x-3)} + \frac{-40}{(x-4)}$$

3] $$\frac{1}{(x^2+1)} + \frac{x}{(x^2+2)}$$

4] $$\frac{5}{(x+1)} + \frac{x-1}{(x^2+x+1)} + \frac{3x+2}{(x^2+x+1)^2}$$

5] $$\frac{-1}{x+1} + \frac{1}{(x-2)} + \frac{6}{(x-2)^2} + \frac{12}{(x-2)^3} + \frac{24}{(x-2)^4}$$

6] $$\frac{3}{(x-1)} + \frac{1}{(x-1)^2} + \frac{7}{(x-1)^3} + \frac{5}{(x-1)^4}$$

7] $$\frac{2}{x-1} + \frac{1}{x-2} + \frac{-1}{(x-2)^2} + \frac{x+1}{x^2+1.}$$

8] $$\frac{-7/16}{x^2+1} + \frac{1/8}{(x^2+1)^2} + \frac{-7/16}{x^2+3} + \frac{3/4}{(x^2+3)^2} + \frac{9/4}{(x^2+3)^3}$$

9] $$\frac{1}{2(x+)} + \frac{-x^3+x^2+3x-4}{2(x^4+3x^3-x^2+4)}$$

10] $$\frac{1}{x} + \frac{-2}{x+1} + \frac{3}{(x+1)^2} + \frac{2x}{x^2-x+1}$$

11] $$\frac{2}{x} + \frac{1}{x^2} + \frac{2}{x^3} - \frac{2x+1}{x^2+1}$$

(284)

$$\frac{4x+1}{x^2+1}+\frac{3x}{(x^2+1)^2}$$

12] $$\frac{-11/8}{(2x+1)}+\frac{15/8}{(2x+5)}$$

❖ ❖ ❖

ERRATA

Page	Line	Incorrect	Correct
9	1	aprocedure	a procedure
10	14	97 + 12 = 85	$97 + \overline{12} = 85$
14	16	88 + 03 = 85	$88 + \overline{03} = 85$
	16	35	3 x 5
15	9	07	$\overline{0\,7}$
	10 & 11	05	$\overline{0\,5}$
	12	7 x 5	$\overline{7} \times \overline{5}$
17	4	(9 - 8)	(9 - 7)
18	12	07904 + 24320	07904 24320
40	10	91|27|27	$91 \mid 27 \mid \overline{27}$
63	7	293 2896	29|32896
64	2	as above	as above
65	6	59 1 91573	59|191573
	18	1797145501	179|7145501
66	2 to 7	1 18 + 0 etc.	1 x 18 + 0 etc.
	18	6119581	61|19581
67	6	19581	$1\,\overline{9}\,5\,\overline{8}\,1$
	18	6710171203	67|10171203
	18	1 0 1 7 1 2 0 3	$\overline{1}\,0\,\overline{1}\,7\,\overline{1}\,2\,0\,3$
75	Last Line	$\frac{611}{4}$	611 4
76	First Line	20 20	$\frac{20}{20}$ (& Shift to Previous Page)
77	Last Line	Osculator digit	Osculater x digit

72	17 to 21	69 5 + 4	69 x 5 + 4
		87 5 + 3 + 8	87 x 5 + 3 + 8
		25 5	25 x 5
		86 5	86 x 5
		97 5	97 x 5
15	9 & 10	07	$\overline{07}$
		05	$\overline{05}$
15	12	7 x 5	$\overline{7} \times \overline{5}$
145	10	[x 72] [yzx] [zxy]	$[xyz] \neq [yzx] \neq [zxy]$
	13		add "implies" after [x, y, z] = [a, b, c]
146	13	T(A) + T(B)	$T(A) + T(B) \neq T(A+B)$
	16		Add "It is a relation" after operators)
147	last line	also	delete also & add : "In the figure given below"
151	8	[3 12 - 45 , 412 - 35 , 513]	[3x12 - 4 x 5, 4 x 12 - 3x5, 5 x 13]
	9	[3 12 + 45, 412 + 35 , 513]	3 x 12 + 4 x 5, 4 x 12 + 3 x 5, 5 x 13]
	19	$[x_2 y_2 z_2][x_1 y_1 z_1]$	$[x_2 y_2 z_2] = [x_1 y_1 z_1]$
154	Last Line	(1 + 2 3...)	$[1 + 2\sqrt{3}]$
	--"--	[3 1 2]	$[\sqrt{3}, 1, 2]$
	--"--	dividing by 3	dividing by $\sqrt{3}$
155	10	$[1-3\ 1+3\ 2\sqrt{2}]$	$[1-\sqrt{3}, 1+\sqrt{3}, 2\sqrt{2}]$
159	2	Sin = 3/2	$\sin\Theta = \sqrt{3}/2$
	3	TO = (x 32)	$T(\Theta) = [x, \sqrt{3}, 2]$
	6	TO = [-1 32]	$T(\Theta) = [-1\ \sqrt{3}\ 2]$
	7	and cos	and cos Θ
	7	tan = - 3	$\tan\Theta = -\sqrt{3}$
	10	TO	$T(\Theta)$

नचिकेत प्रकाशन ग्रंथसूची २०१४

यशस्वी द्विशतकी वाटचाल

✠ वैज्ञानिक शिल्पकार : डॉ. होमी भाभा	किं. १००
✠ सार्वजनिक ग्रंथालय मार्गदर्शक	किं. २००
✠ ग्राहक चेतना	किं. २००
✠ मराठी ग्रंथसंपदा	किं. २००
✠ ओंजळीतील मोती	किं. १३०
✠ सगे सोयरे	किं. २००
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✠ काव्यप्रभा	किं. ९०
✠ चिंतन मंथन	किं. ७५
✠ योगिनी	किं. ८०
✠ युगाब्द तिथी दैनंदिनी ५११४	किं. २०

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✠ देवस्वरूपा कामधेनु : वैज्ञानिक महत्व (दु.आ.)	किं. १६०
✠ पंचगव्य औषधोपचार (दु.आ.)	किं. १००
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✠ यशस्वी दुकानदारी (दु.आ.)	किं. १६०

बँकिंग व कायदा प्रकाशने

✠ बहुराज्यीय सहकारी संस्था मार्गदर्शक	किं २५०
✠ सहकारी वित्तीय संस्था निवडणूक मार्गदर्शक	किं ३५०
✠ सहकारी वित्तीय संस्था कनिष्ठ श्रेणी सेवक मार्गदर्शक	किं १५०
✠ महाराष्ट्र सावकारी (नियमन) अधिनियम २०१४	किं १२५
✠ बँकिंग व्यवसाय धोरणे (दुसरी आवृत्ती)	किं. ६५०
✠ बँकिंग रेग्युलेशन ॲक्ट मराठी अनुवादासह	किं. ५५०
✠ अंतर्गत नियंत्रण व्यवस्थापन	किं. ३५०
✠ मल्टिस्टेट को-ऑप. ॲक्ट	किं. ५५०
✠ मल्टिस्टेट सोसा. आदर्श पोटनियम	किं. १५०
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✠ बँकिंग धोरणे	किं. ५००
✠ बँकिंग परिभाषा कोश	किं. ४००
✠ बँकिंग प्रश्नोत्तरे	किं. ३००
✠ निवडक बँकिंग निवाडे	किं. ३५०
✠ बँकिंग निवाडे डायजेस्ट	किं. ५००
✠ पतसंस्था व्यवस्थापन	किं. ३५०
✠ शाखा व्यवस्थापन (दु.आ.)	किं. २५०
✠ कर्मचारी व्यवस्थापन	किं. २५०
✠ सीईओ : भूमिका व जबाबदारी	किं. २५०
✠ ना. बँकासाठी सहकारी परिपत्रके	किं. ३५०
✠ पतसंस्थासाठी सहकारी परिपत्रके	किं. ३५०
✠ नागरी बँक संदर्भ (२३ वी आवृत्ती)	किं. ५००
✠ पतसंस्था संदर्भ (२३ वी आवृत्ती)	किं. ४५०
✠ संचालक मार्गदर्शक (५वी आवृत्ती)	किं. ४५०
✠ कर्जवसुली मार्गदर्शक (५ वी आवृत्ती)	किं. ४५०
✠ कर्मचारी सेवा पुस्तिका	किं. २५०
✠ किंमत विश्लेषण आणि ताळेबंद विश्लेषण	किं. १५०
✠ सहकारी परिपत्रके २००८ ते १२	किं. ३५०
✠ ऑडीट मार्गदर्शक	किं. ५५०
✠ रिझर्व बँक मास्टर परिपत्रके	किं. १५००
✠ आर्बिटेशन ॲक्ट-लवाद कायदा	किं. ३००
✠ व्यवसाय व्यवस्थापन	किं २५०
✠ पतसंस्था धोरणे	किं ३५०
✠ लेखा परीक्षण आणि सभा व्यवस्थापन	किं ३००
✠ कुंडली कर्ज व्यवहाराची	किं ५५०